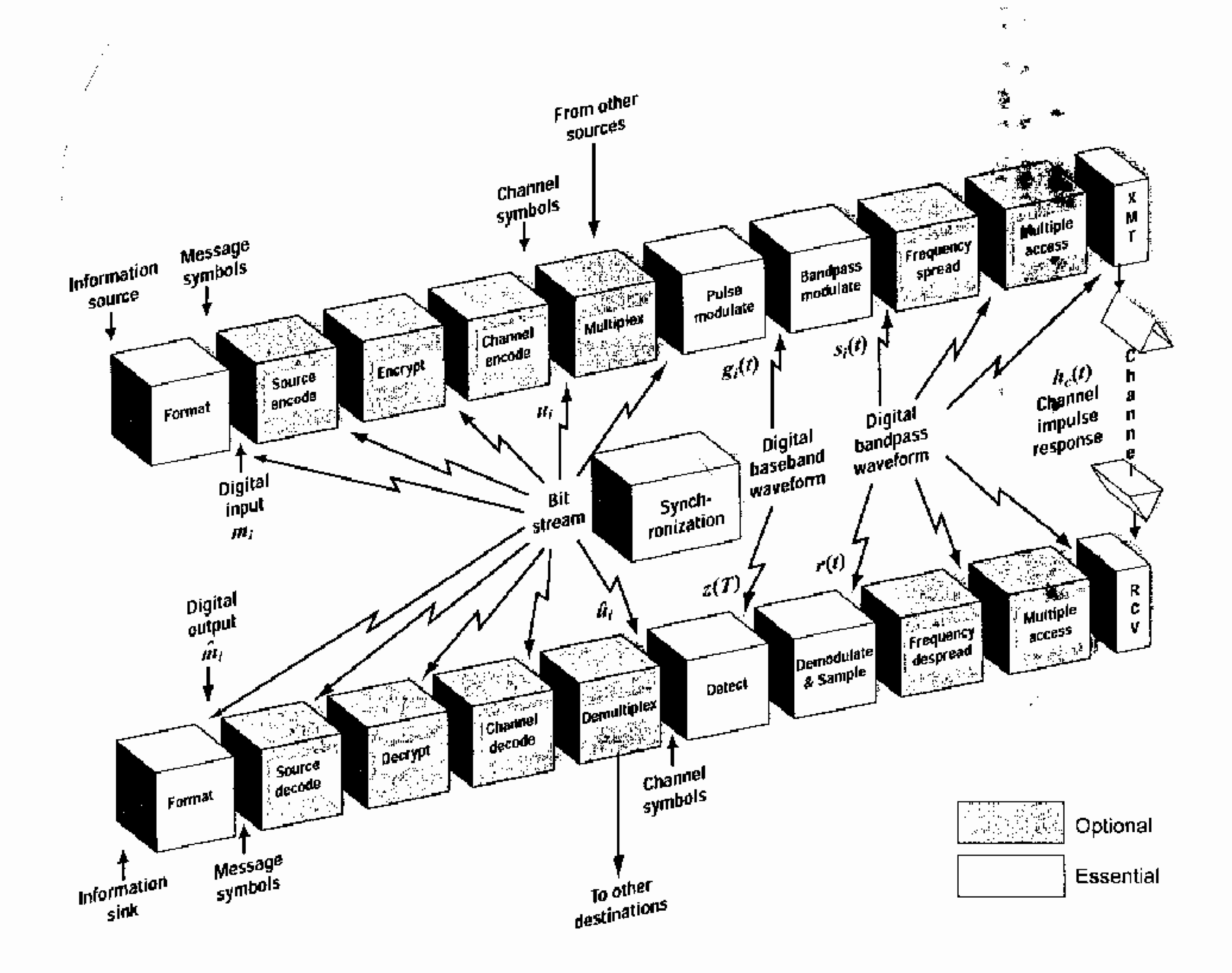
Solutions Manual for

Digital Communications

SECOND EDITION

Fundamentals and Applications



BERNARD SKLAR

Solutions Manual to Accompany

Digital Communications
Second Edition

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Chapter 1

1.1 (a) $x(t) = A\cos 2\pi f_0 t$: Power signal

$$P_x = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x^2(t)\,dt = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} A^2\cos^2 2\pi f_0 t\,dt$$

$$= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \frac{A^2}{2}\,dt = \frac{A^2}{2}$$

(b)

$$x(t) = \begin{cases} A\cos 2\pi f_0 t & \text{for } -T_0/2 \le t \le \frac{T_0}{2} \\ 0 & \text{elsewhere} \end{cases}$$

Energy signal

$$E_x = \int_{-\infty}^{\infty} x^2(t)\,dt = \int_{-T_0/2}^{T_0/2} A^2\cos^2 2\pi f_0 t\,dt = \frac{A^2 T_0}{2}$$

(c)

$$x(t) = \begin{cases} A\exp(-at) & \text{for } t > 0,\ a > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Energy signal

$$E_x = \int_{-\infty}^{\infty} x^2(t)\,dt = \int_{0}^{\infty} A^2\exp(-2at)\,dt$$

$$= \left[\frac{A^2\exp(-2at)}{-2a}\right]_0^{\infty} = \frac{A^2}{2a}$$

(d) $x(t) = \cos t + 5\cos 2t \quad \text{for } -\infty < t < \infty$

Power signal

$$P_x = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \cos^2 t + 25\cos^2 2t \, dt; \quad 2\pi f_0 = 1, \; T_0 = 2\pi$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{1}{2} + \frac{25}{2}\right) dt = \frac{1}{2\pi}(26\pi) = 13$$

1.2 $x(t) = \text{rect}(t/T)$

$$= \begin{cases} 1 & \text{for } -T/2 \le t \le T/2 \\ 0 & \text{elsewhere} \end{cases}$$

ESD $\Psi(f) = |X(f)|^2 = T^2 \,\text{sinc}^2(fT)$

$$E_x = \int_{-\infty}^{\infty} x^2(t)\, dt = \int_{-T/2}^{T/2} dt = T$$

1.3 Using Equations (1.18) and (1.19)

$$G_x(f) = \sum_{n=-\infty}^{\infty} |c_n|^2 \,\delta(f - nf_0)$$

$$P_x = \int_{-\infty}^{\infty} G_x(f)\, df = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} |c_n|^2 \delta(f - nf_0)\, df$$

$$P_x = \sum_{n=-\infty}^{\infty} |c_n|^2$$

1.4 $P_x = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x^2(t)\,dt\,; \qquad 2\pi f_0 = 10$

$f_0 = 5/\pi$

$T_0 = \pi/5$

$$P_x = \frac{5}{\pi}\int_{-\pi/10}^{\pi/10} 100\cos^2 10t + 400\cos^2 20t\,dt$$

$$= \frac{5}{2\pi}\int_{-\pi/10}^{\pi/10} 100(1+\cos 20t) + 400(1+\cos 40t)\,dt$$

$$= \frac{5}{2\pi}\Big[100t + 400t\Big]_{-\pi/10}^{\pi/10} = 250 \text{ W}$$

1.5 $G_x(f) = \sum_{n=-\infty}^{\infty} |c_n|^2\,\delta(f - nf_0)$

$c_1 = c_{-1} = \frac{10}{2} = 5; \quad c_2 = c_{-2} = \frac{20}{2} = 10$

$c_n = 0$ for $n = 0, \pm 3, \pm 4, \cdots$

$$G_x(f) = (5)^2\,\delta\left(f - \tfrac{5}{\pi}\right) + (5)^2\,\delta\left(f + 5/\pi\right)$$
$$+ (10)^2\,\delta\left(f - 10/\pi\right) + (10)^2\,\delta\left(f + 10/\pi\right)$$

$$P_x = \int_{-\infty}^{\infty} G_x(f)\,df = 25 + 25 + 100 + 100$$
$$= 250 \text{ W}$$

1.6 $\mathcal{F}\{R(\tau)\}$ must be a nonnegative function because $\mathcal{F}\{R(\tau)\} = G(f)$; and, the power spectral density, $G(f)$, must be a nonnegative function.

(a) $x(\tau) = \begin{cases} 1 & \text{for } -1 \le \tau \le 1 \\ 0 & \text{otherwise} \end{cases}$

[Sketch: rectangular pulse of height 1 from -1 to 1 on the τ axis]

NO
$\begin{cases} 1.\ x(\tau) = x(-\tau)\ ✓ \\ 2.\ x(0) \ge x(\tau)\ ✓ \\ 3.\ \mathcal{F}\{x(\tau)\} \text{ is a positive } \underline{\text{and}} \ \underline{\text{negative}} \text{ going function.} \end{cases}$

(b) $x(\tau) = \delta(\tau) + \sin 2\pi f_0 \tau$

NO 1. $x(\tau) \ne x(-\tau)$ ×

(c) $x(t) = \exp(|\tau|)$

NO
1. $x(\tau) = x(-\tau)$ ✓
2. $x(0) \not\ge x(\tau)$ ×

(d) $x(\tau) = \begin{cases} -\tau + 1 & \text{for } 0 \le \tau \le 1 \\ \tau + 1 & \text{for } -1 \le \tau \le 0 \end{cases}$

[Sketch: triangle of height 1 from -1 to 1 on the τ axis]

YES
$\begin{cases} 1.\ x(\tau) = x(-\tau)\ ✓ \\ 2.\ x(0) \ge x(\tau)\ ✓ \\ 3.\ \mathcal{F}\{x(\tau)\} = 2\,\text{sinc}^2 f\tau \end{cases}$

is a nonnegative function. ✓

1.7 (a) $X(f) = \delta(f) + \cos^2 2\pi f$

YES
1. always real ✓
2. $P_x(f) \geq 0$ ✓
3. $P_x(-f) = P_x(f)$ ✓

(b) $X(f) = 10 + \delta(f-10)$

No
1. always real ✓
2. $P_x(f) \geq 0$ ✓
3. $P_x(-f) \neq P_x(f)$ ×

(c) $X(f) = \exp(-2\pi|f-10|)$

No
1. always real ✓
2. $P_x(f) \geq 0$ ✓
3 $P_x(-f) \neq P_x(f)$ ×

(d) $X(f) = \exp[-2\pi(f^2-10)]$

YES
1. always real ✓
2. $P_x(f) \geq 0$ ✓
3. $P_x(-f) = P_x(f)$ ✓

1.8

$$R_x(\tau) = \left\langle A\cos(2\pi f_0 t+\phi)\, A\cos(2\pi f_0 t + 2\pi f_0 \tau + \phi)\right\rangle$$

where $\langle\cdot\rangle$ is the time averaging operator $\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} dt$

Upon expanding (see Appendix D), $R_x(\tau)$ becomes:

$$R_x(\tau) = A^2\Big[\cos 2\pi f_0\tau \left\langle \cos^2(2\pi f_0 t+\phi)\right\rangle$$

$$- \sin 2\pi f_0 \tau \left\langle \cos(2\pi f_0 t+\phi)\sin(2\pi f_0 t+\phi)\right\rangle\Big]$$

The negative term in the above expression goes to zero, and hence

$$R_x(\tau) = \frac{A^2}{2}\cos 2\pi f_0 \tau$$

$$P_x = R_x(0) = A^2/2$$

1.9 (a) $R_x(\tau) = \frac{100}{2}\cos 10\tau + \frac{400}{2}\cos 20\tau$

where $2\pi f_0 = 10$

(b) $P_x = R_x(0) = 50 + 200 = 250\ \text{W}$

1.10 (a) average value of $x(t)$

$$\langle x(t) \rangle = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} (1 + \cos 2\pi f_0 t)\, dt = 1$$

(b) the ac power of $x(t)$

$$\langle \sigma^2(t) \rangle = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \cos^2 2\pi f_0 t \, dt = \frac{1}{2}$$

(c) the rms value of $x(t)$

$$\langle x^2(t) \rangle = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} (1 + \cos 2\pi f_0 t)^2\, dt$$

$$= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} (1 + 2\cos 2\pi f_0 t + \cos^2 2\pi f_0 t)\, dt = \frac{3}{2}$$

$$x_{rms} = \sqrt{3/2}$$

1.11 (a) $$\langle X(t) \rangle = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} A \cos(2\pi f_0 t + \phi)\, dt = 0$$

$$\langle X^2(t) \rangle = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} [A \cos(2\pi f_0 t + \phi)]^2\, dt$$

$$= \frac{A^2}{2}$$

1.11 (b) $E\{X\} = \int_{-\infty}^{\infty} X(\phi)\, p(\phi)\, d\phi$

$p(\phi) = \frac{1}{2\pi}$ since ϕ is uniformly distributed over $(0, 2\pi)$

$$E\{X\} = \int_0^{2\pi} \left[A \cos(2\pi f_0 t + \phi)\right] \frac{1}{2\pi}\, d\phi = 0$$

$$E\{X^2\} = \int_0^{2\pi} \left[A \cos(2\pi f_0 t + \phi)\right]^2 \frac{1}{2\pi}\, d\phi$$

$$= A^2/2$$

1.12 $X(f) = \text{sinc } f$

$$\Psi_x(f) = |X(f)|^2 = \text{sinc}^2 f$$

$$R_x(\tau) = \mathcal{F}^{-1}\{\Psi_x(f)\}$$

From Table A.1, $R_x(\tau)$ is seen to be the following triangular function:

$$R_x(\tau) = \begin{cases} 1 - |\tau| & \text{for } |\tau| < 1 \\ 0 & \text{elsewhere} \end{cases}$$

1.13 (a) $\int_{-\infty}^{\infty} \cos 6t \, \delta(t-3)\, dt = \cos 18$

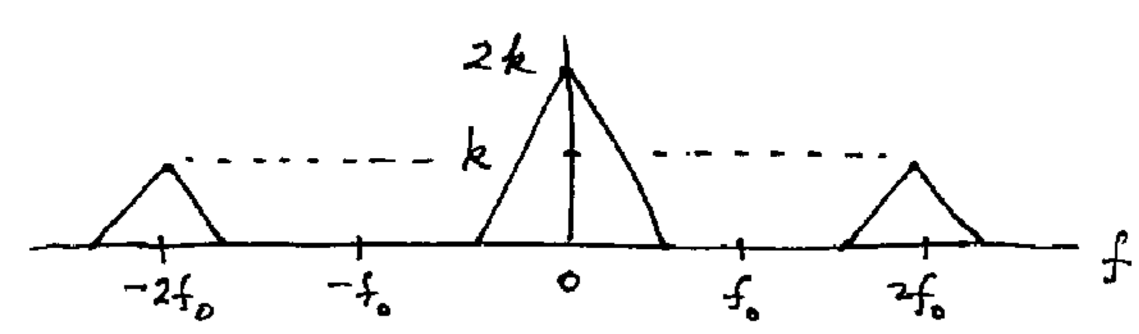
(b) $\int_{-\infty}^{\infty} 10\, \delta(t)\, (1+t)^{-1}\, dt = 10$

(c) $\int_{-\infty}^{\infty} \delta(t+4)\, (t^2+6t+1)\, dt = -7$

(d) $\int_{-\infty}^{\infty} \exp(-t^2)\, \delta(t-2)\, dt = 0.0183$

1.14 $X_2(f) = k\left[\delta(f-f_0) + \delta(f+f_0)\right]$

$X_1(f) * X_2(f) = X_1(f) * k\left[\delta(f-f_0) + \delta(f+f_0)\right]$



1.15 (a) $P_x = 2\int_0^{10\text{ kHz}} G_x\, df = 2\int_0^{10\text{ kHz}} 10^{-6} f^2\, df$

$= 2\left[\frac{10^{-6} f^3}{3}\right]_0^{10^4} = 667 \text{ kW}$

(b) $P_x = 2\int_{5\text{ kHz}}^{10\text{ kHz}} 10^{-6} f^2\, df = 2\left[\frac{10^{-6} f^3}{3}\right]_{5000}^{10000}$

$= 583 \text{ kW}$

1.16 $10 \log_{10}\left[\frac{100 \times 2 \times \frac{1}{2}}{\frac{1}{2}}\right] = 23\,dB$

1.17 (a) Since $|H(f)|$ decreases monotonically with $|f|$, and $|H(0)| = 1$, we can write the following relationship in terms of the $-1\,dB$ frequency, f_1.

$$10 \log_{10} |H(f_1)|^2 = -1\,dB$$

$$\log_{10}\left[\frac{1}{(1 + f_1/f_u)^{2n}}\right] = -\frac{1}{10}$$

$$\left[1 + f_1/f_u\right]^{2n} = 10^{1/10}$$

$$\left[f_1/f_u\right]^{2n} = 10^{1/10} - 1 = 0.2584$$

$$\therefore\ n \geq \frac{1}{2}\left[\frac{\log 0.2584}{\log (f_1/f_u)}\right]$$

For $f_1/f_u = 0.9$, $n \geq 6.4$

Thus, $n = 7$

1.17 (b) In the limit, as $n \to \infty$

$\left(\frac{f}{f_u}\right)^{2n} \to 0$, $|H(f)| \to 1$, for $\left|\frac{f}{f_u}\right| < 1$

$\left(\frac{f}{f_u}\right)^{2n} \to \infty$, $|H(f)| \to 0$, for $\left|\frac{f}{f_u}\right| > 1$

Hence, $|H(f)|$ approaches the transfer characteristic of an ideal low-pass filter with a cut-off frequency at f_u, as n approaches infinity.

1.18 $y(t) = \delta(t) * h(t)$

$$Y(f) = 1 \times H(f)$$

since $\mathcal{F}\{\delta(t)\} = 1$.

Hence, $y(t) = \mathcal{F}^{-1}\{Y(f)\} = \mathcal{F}^{-1}\{H(f)\}$

$$= h(t)$$

1.19 Let $x(t) = \delta(t)$

$$g(t) = \delta(t) - \delta(t - t_0)$$

$$h(t) = \int_{-\infty}^{t} [\delta(\tau) - \delta(\tau - t_0)]\, d\tau$$

$$= \mu(t) - \mu(t - t_0)$$

where $\mu(t)$ is the unit step function defined as follows:

$$\mu(t) = \begin{cases} 1 & t > 0 \\ 0 & \text{elsewhere} \end{cases}$$

$h(t)$
1
t
t_0

1.20 (a) Half-power bandwidth is the bandwidth from half-power point to half-power point. BW = $2f_0$

where $$\frac{1}{2} = \left[\frac{\sin(\pi f_0 10^{-4})}{\pi f_0 10^{-4}}\right]^2$$

$$0.707 = \frac{\sin x_0}{x_0}, \quad x_0 = \pi f_0 10^{-4}$$

$x_0 \cong 1.4 \Rightarrow f_0 = 4.46$ kHz $\Rightarrow$ BW $\cong$ 9 kHz

1.20 (b) Noise equivalent bandwidth

$$BW = 2\int_0^\infty \left[\frac{\sin(\pi f 10^{-4})}{\pi f 10^{-4}}\right]^2 df$$

$$= \frac{2\times 10^4}{\pi}\underbrace{\int_0^\infty \left[\frac{\sin x}{x}\right]^2 dx}_{\pi/2}$$

$$= 10 \text{ kHz}$$

(c) Null-to-null bandwidth: $BW = 2f_0$ where f_0 is the frequency where

$$\frac{\sin(\pi f_0 10^{-4})}{\pi f_0 10^{-4}} = 0$$

The minimum f_0 corresponding to the first null is found by:

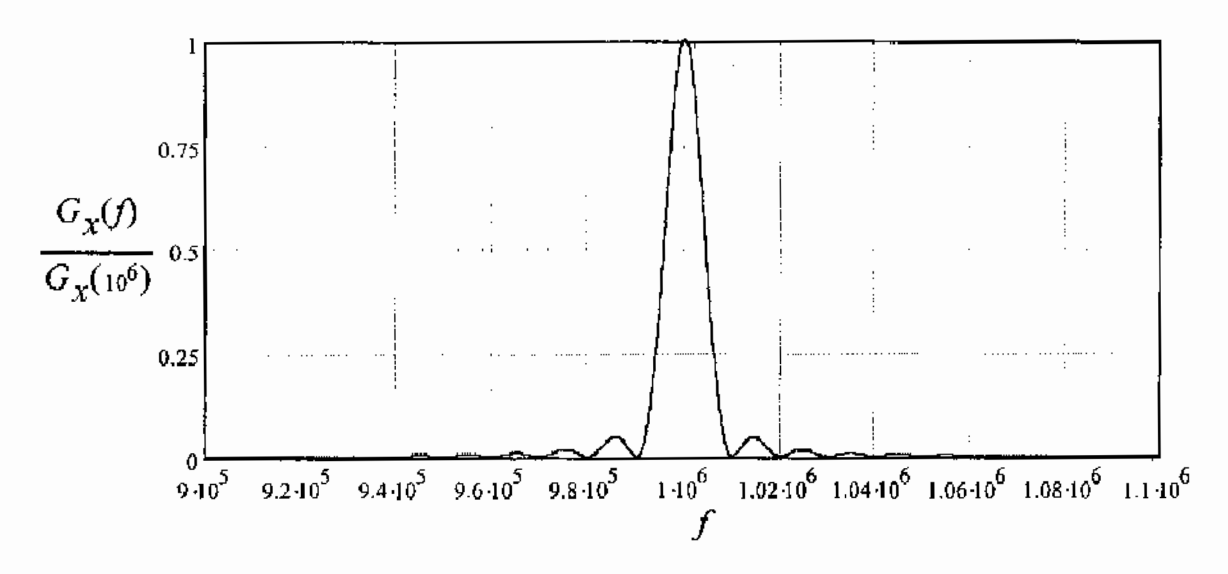
$$\pi f_0 10^{-4} = \pi$$

$$BW = 2f_0 = 20 \text{ kHz}$$

(d) 99% of power bandwidth. $BW = 2f_0$

where
$$0.995 = \frac{10^{-4}\int_0^{f_0}\left(\frac{\sin \pi f 10^{-4}}{\pi f 10^{-4}}\right)^2 df}{10^{-4}\int_0^{\infty}\left(\frac{\sin \pi f 10^{-4}}{\pi f 10^{-4}}\right)^2 df}$$

1.20 (d) The normalized spectrum, $G_x(f)/G_x(10^6)$, appears as:



Applying numerical methods with Mathcad ®, the two-sided 99% bandwidth can be found by numerical integration as:

$$\frac{\int_{10^6-f_1}^{10^6+f_1} G_x(f)\,df}{\int_{-\infty}^{\infty} G_x(f)\,df} = 0.99$$

where f_1 is found to be equal to 103 kHz. Thus, the two-sided 99% bandwidth is equal to 206 kHz.

Since this bandwidth corresponds to the given spectrum (with signaling rate = 10,000 symbols/s), normalizing it relative to one symbol per second, yields the two-sided 99% bandwidth as $(206\times10^3/10^4) = 20.6$ Hz for one symbol per second, or in general the 99% bandwidth in terms of the signaling rate, R, is $20.6\times R$ Hz.

1.20 (e)

35-dB Bandwidth:

35-dB attenuation $\Longrightarrow 10^{-3.5} = 3.16\times10^{-4}$

Since $\sin^2 x$ is unity for $x = \frac{\pi}{2}(2k+1)$, $k = 0, 1, \cdots$, the lobe beyond which the attenuation criterion is guaranteed to be met is the minimum k for which

$$10^{-3.5} \geq \frac{1}{[\pi/2\,(2k+1)]^2}$$

$$3162.28 \leq [\pi/2\,(2k+1)]^2$$

$$56.23 \leq \pi/2\,(2k+1)$$

$$35.80 \leq 2k+1 \implies k = 18$$

Thus the 18th sidelobe meets the 35-dB criterion, and the actual 35-dB point will be on the falling edge of the 17th sidelobe. $BW = 2f_0$, where f_0 is the minimum value satisfying:

$$\pi f_0\, 10^{-4} > \frac{\pi}{2}(35)$$

and $$\left[\frac{\sin(\pi f_0\, 10^{-4})}{\pi f_0\, 10^{-4}}\right]^2 = 10^{-3.5}$$

Solving iteratively, we get: $\pi f_0\, 10^{-4} = 55.171$

$$BW = 2f_0 = 351.2 \text{ kHz}$$

1.20 (f) The absolute bandwidth is <u>infinite</u>, since for any finite test-BW, $10^{-4}\left[\frac{\sin \pi(f-10^6)10^{-4}}{\pi(f-10^6)10^{-4}}\right]^2$ will have positive measure beyond it.

Chapter 2

2.1 (a) 00010010 (H), 11110011 (O), 11101011 (W)

(b) 000, 100, 101, 111, 001, 111, 101, 011 → 0 4 5 7 1 7 5 3

8 symbols

(c) 24 bits / 4 bits per symbol = 6 symbols

(d) 24 bits/8 bits per symbol = 3 symbols

2.2 (a) 800 char/s × 8 bits/char = 6400 bits/s

(b) $\frac{6400 \text{ bits/s}}{4 \text{ bits/symbol}}$ = 1600 symbols/s.

2.3 (a) 100 char/2 s × 8 bits/char = 400 bits/s.

$\frac{400 \text{ bits/s}}{5 \text{ bits/symbol}}$ = 80 symbols/s

(b) 16-level PCM: 400 bits/s, 100 symbols/s.

8-level PCM: 400 bits/s, 133.3 symbols/s.

4-level PCM: 400 bits/s, 200 symbols/s.

2-level PCM: 400 bits/s, 400 symbols/s.

2.4 $x_s(t) = x(t)\, x_p(t)$

$$= x(t)\left\{\sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_s t}\right\}$$

$$= x(t)\left\{c_0 + 2\sum_{n=1}^{\infty} c_n \cos 2\pi n f_s t\right\}$$

$$x_1(t) = x_s(t)\cos 2\pi m f_s t$$

$$= x(t)\left\{\begin{array}{l} c_0 \cos 2\pi m f_s t \quad \text{(LPF)} \\ + 2\sum_{\substack{n=1 \\ n\neq m}}^{\infty} c_n \cos 2\pi n f_s t \cos 2\pi m f_s t \quad \text{(LPF)} \\ + 2 c_m \cos^2 2\pi m f_s t \end{array}\right\}$$

$$x_o(t) = x(t)\; 2c_m\left(\frac{1}{2} + \frac{1}{2}\cos 4\pi m f_s t\right) \quad \text{(LPF)}$$

$$= c_m\, x(t)$$

2.5 (a) L quantization levels requires a minimum of $\log_2 L$ bits. It is necessary to transmit at least $\log_2 L$ bits in T_s seconds. Thus, the time duration, T, for one bit is

$$T \leq \frac{T_s}{\log_2 L}$$

(b) The equality sign is valid if L is a power of 2.

2.6 (a) $2^5 = 32$ (b) $2^8 = 256$ (c) 2^x

2.7

$$x(t) = \frac{\sin 6280t}{6280\,t} = \frac{\sin \frac{Wt}{2}}{Wt/2}$$

where $\frac{W}{2} = 2\pi f = 6280$ radians

$f = 1000$ Hz

$X(f)$: $1/W$ for -1000 Hz ($-\frac{W}{2}$) to 1000 Hz ($\frac{W}{2}$)

$$X(f) = \begin{cases} 1/W & \text{for } |f| \le 1000 \text{ Hz} \\ 0 & \text{elsewhere} \end{cases}$$

$f_m = 1000$ Hz

Minimum sampling rate $f_s = 2 f_m$

$= 2000$ samples/s.

2.8 (a) $\left(\frac{S}{N}\right)_q = 3L^2 \geq 30\text{ dB}$

$$10 \log_{10}(3L^2) \geq 30\text{ dB}$$

$$L = \lceil 18.26 \rceil = 19$$

minimum number of quantization levels

$$\ell = \lceil \log_2 L \rceil = \lceil \log_2 19 \rceil = 5 \text{ bits/sample}$$

(b) $T_b = \frac{T_s}{\ell} = \frac{1}{\ell f_s} = \frac{1}{5(8000)} = 25\,\mu s.$

where T_b is the time duration of a bit.

Required bandwidth, W, is

$$W = \frac{1}{T_b} = \frac{1}{25\,\mu s} = 400\text{ kHz}.$$

2.9 (a) $\omega_m = 2\pi f_m = 2000$

$$f_m = 2000/2\pi = 318.3\text{ Hz}$$

$f_s \geq 2 f_m = 636.6$ samples/s.

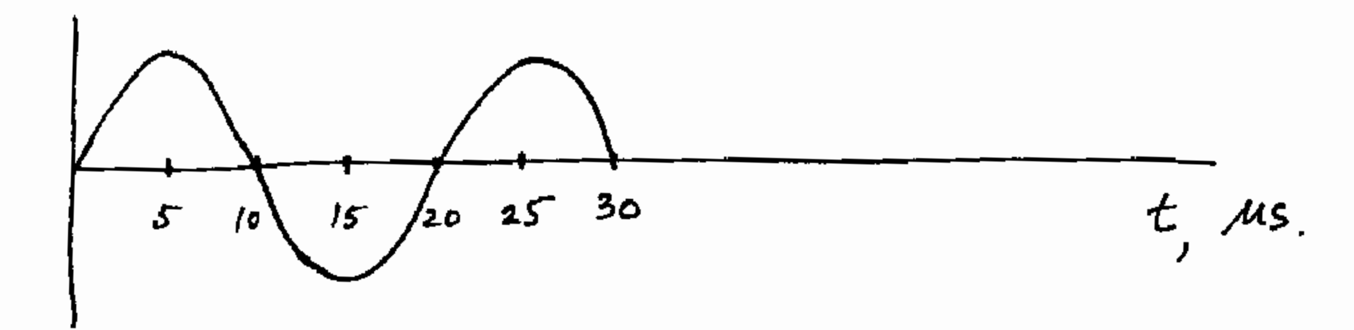
$T_s = \frac{1}{f_s} \leq 0.00157$ s.

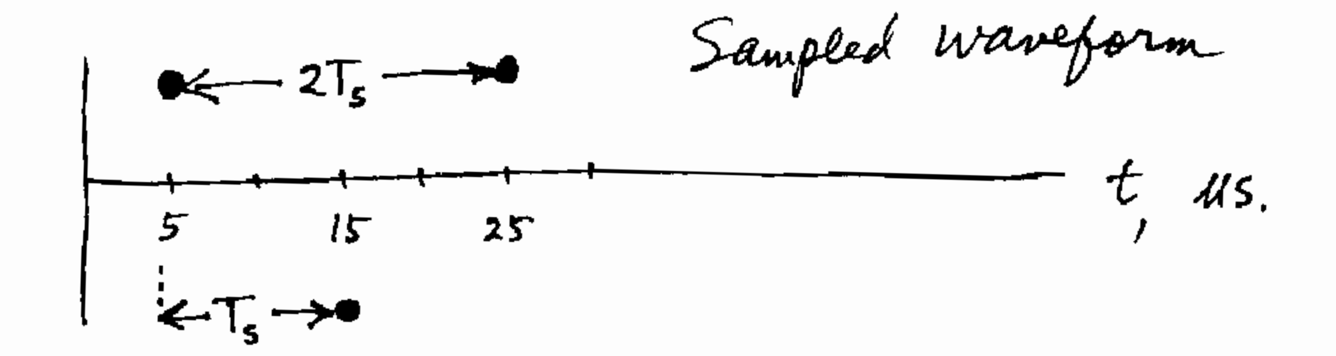
(b) 636.6 samples/s $\times$ 3600 s

$= 2.29 \times 10^6$

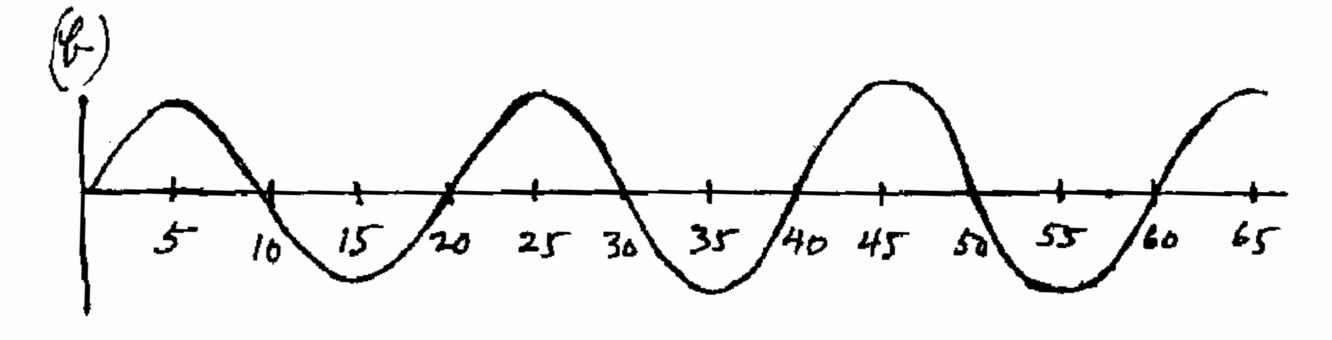
samples

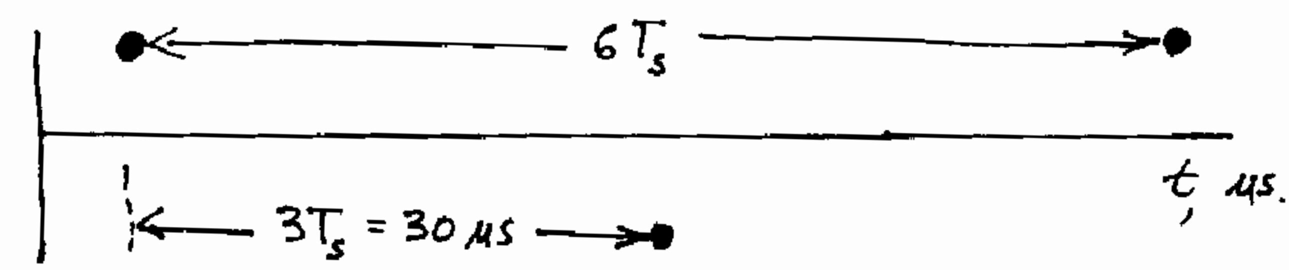
2.10 (a) $f_o = 50\ kHz$; $T_s = \frac{1}{2f_o} = 10\,\mu s.$





(b)

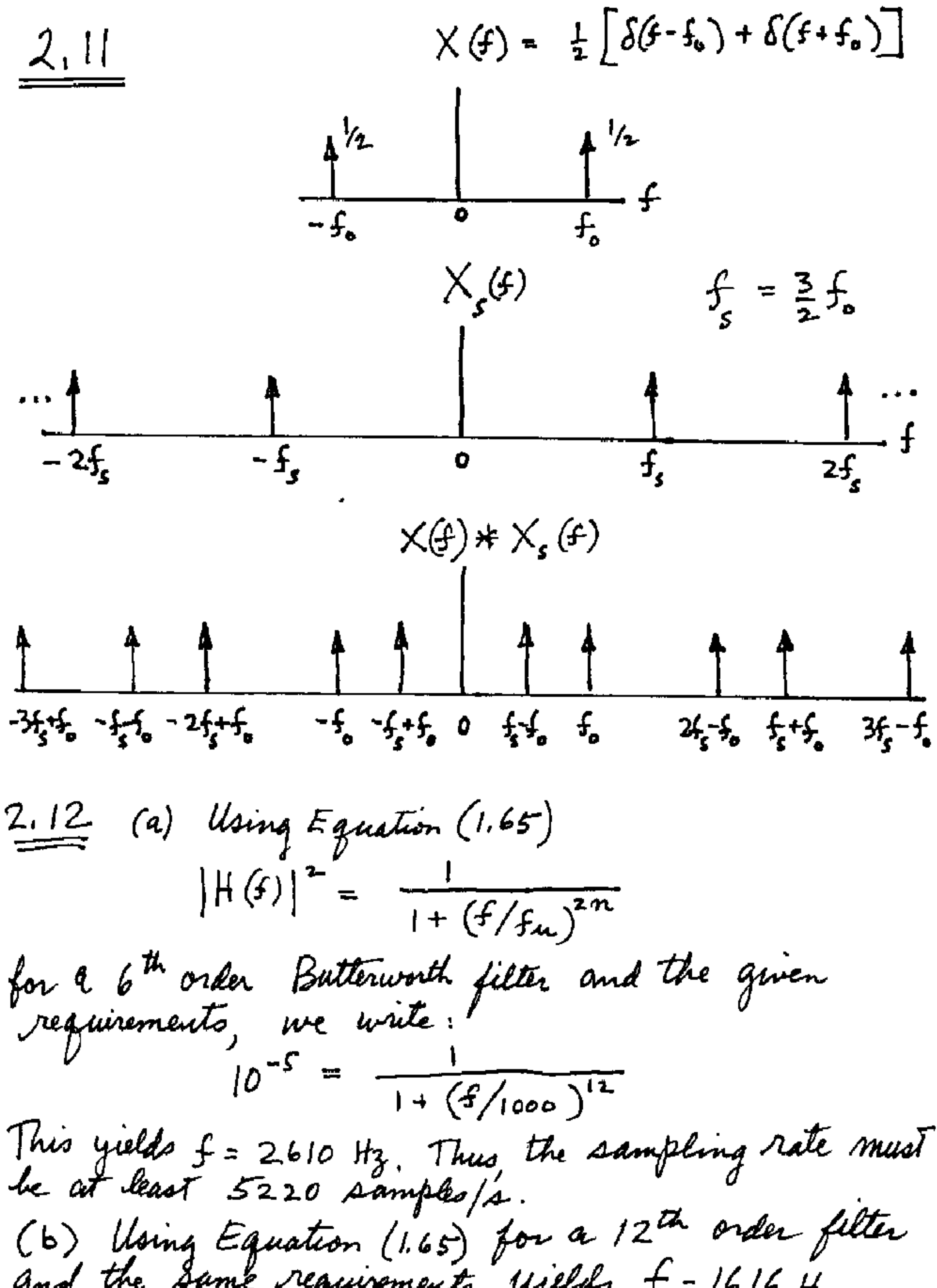




Period of reconstructed waveform $= 6T_s$

$\therefore\ f = \frac{1}{6T_s} = 16.67\ kHz \neq 50\ kHz.$

2.11



$$X(f) = \frac{1}{2}\left[\delta(f-f_0) + \delta(f+f_0)\right]$$

$$f_s = \frac{3}{2} f_0$$

2.12

(a) Using Equation (1.65)

$$|H(f)|^2 = \frac{1}{1 + (f/f_u)^{2n}}$$

for a 6^{th} order Butterworth filter and the given requirements, we write:

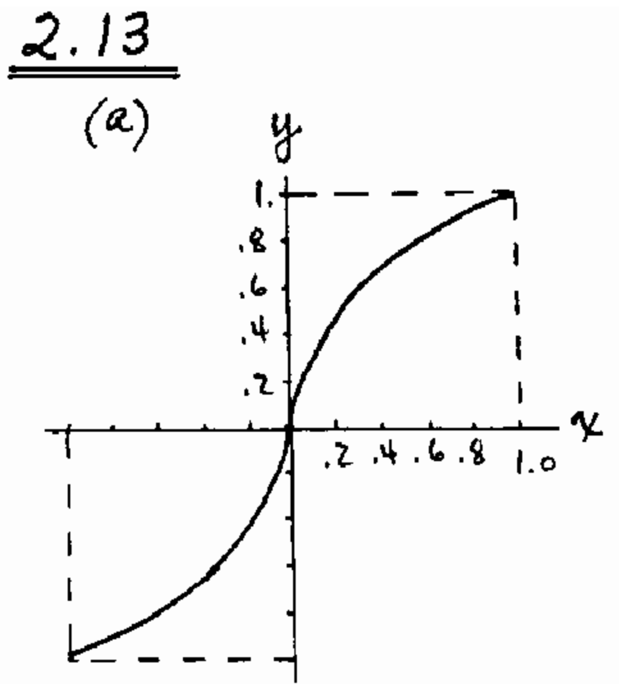
$$10^{-5} = \frac{1}{1 + (f/1000)^{12}}$$

This yields $f = 2610$ Hz. Thus, the sampling rate must be at least 5220 samples/s.

(b) Using Equation (1.65) for a 12^{th} order filter and the same requirements yields $f = 1616$ Hz. Thus, the sampling rate must be at least 3232 samples/s.

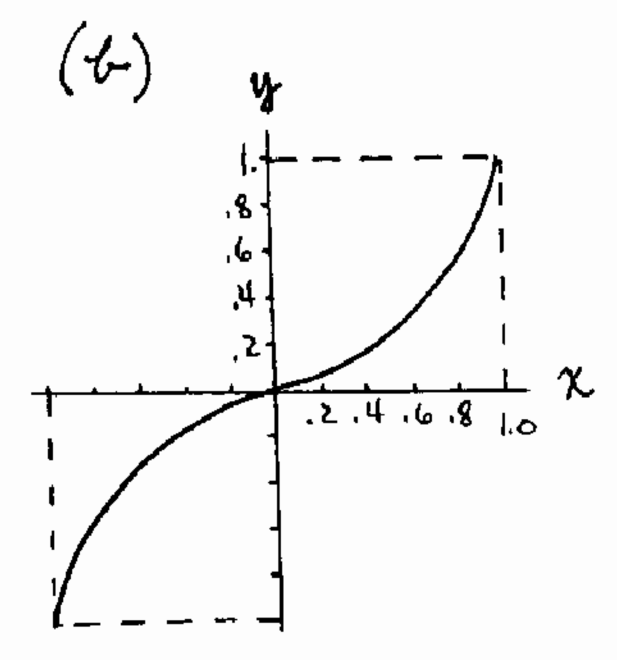
2.13

(a)



compression characteristic $\mu = 10$

(b)



expansion characteristic $\mu = 10$

(c)

16-level nonuniform quantizer Characteristic

2.14 (a) $L \geq \frac{1}{2p} = \frac{1}{0.02} = 50$ levels

$\ell = \lceil \log_2 50 \rceil = 6$ bits/sample

(b) $f_s = 2f_m = 2 \times 4000 = 8000$ samples/s.

Bit rate: $R = 8000$ samples/s $\times$ 6 bits/sam
$= 48{,}000$ bits/s.

(c) 16-level pulses: $16 = M = 2^k$

$k = 4$ bits/pulse

Symbol rate: $R_s = \frac{R}{\log_2 M} = \frac{48{,}000 \text{ bits/s}}{4 \text{ bits/symbol}}$

$= 12{,}000$ symbols/s.

2.15 Binary case:

$R = 8000$ samples/s $\times$ 6 bits/sample $= 48{,}000$ bits/s

$W = \frac{1}{T_b} = R = 48{,}000$ Hz.

$\left(\frac{S}{N}\right)_q = 3L^2 = 3(64)^2 = 12{,}288$

$\cong 41$ dB

Four-level case:

$R_s = \frac{48{,}000 \text{ bits/s}}{2 \text{ bits/symbol}} = 24{,}000$ symbols/s.

$W = \frac{1}{T} = R_s = 24{,}000$ Hz

$\left(\frac{S}{N}\right)_q$ = the same as in the binary case $\cong 41$ dB

2.16 (a) Assume that the L quantization levels are equally spaced and symmetrical about zero. Then, the maximum possible quantization noise voltage equals ½ the q volt interval between any two neighboring levels, Also, the peak quantization noise power, N_q, can be expressed as $(q/2)^2$.

The peak signal power, S, can be designated $(V_{pp}/2)^2$, where $V_{pp} = V_p - (-V_p)$ is the peak-to-peak signal voltage, and V_p is the peak voltage.

Since there are L quantization levels and $(L - 1)$ intervals (each interval corresponding to q volts), we can write:

$$\left(\frac{S}{N_q}\right)_{\text{peak}} = \frac{(V_{pp}/2)^2}{(q/2)^2} = \frac{\left[q(L-1)/2\right]^2}{(q/2)^2}$$

$$\approx \frac{q^2L^2/4}{q^2/4} = L^2$$

Thus, we need to compute how many levels, L, will yield a $(S/N_q)_{\text{peak}} = 96$ dB. We therefore write:

$$96\,\text{dB} = 10\log_{10}\left(\frac{S}{N_q}\right)_{\text{peak}} = 10\log_{10} L^2$$

$$= 20\log_{10} L$$

$$L = 10^{96/20} = 63{,}096 \quad \text{levels}$$

(b) The number of bits that correspond to 63,096 levels is

$$\ell = \lceil \log_2 L \rceil = \lceil \log_2 63{,}096 \rceil = 16 \quad \text{bits/sample}$$

(c) $R = 16$ bits/sample × 44.1 ksamples/s = 705,600 bits/s.

2.17 Let the peak-to-peak amplitude separation be A volts.

Bipolar case (NRZ):

+A/2 binary one, 0, −A/2 binary zero, t

Average power =

$$\frac{1}{2}\left(\frac{A}{2}\right)^2 + \frac{1}{2}\left(\frac{A}{2}\right)^2 = \frac{A^2}{4}$$

Unipolar case (RZ):

A binary one, 0, binary zero, t

Average power =

$$\frac{1}{2}\left(A^2\right) + \frac{1}{2}\left(0\right)^2$$

$$= \frac{A^2}{2}$$

Bipolar signaling requires half the average power for the same separation between the binary one and zero. The disadvantage in using bipolar signaling is the need for 2 power supplies.

2.18 The data rate for T1 service is:

24 samples/frame × 8 bits/sample × 8000 frames/s
+1 alignment bit/frame
= 193 bits/frame × 8000 frames/s = 1.544×10^6 bits/s

Bandwidth efficiency is:

$$\frac{R}{W} = \frac{1.544 \times 10^6}{386 \times 10^3} = 4 \text{ bits/s/Hz}.$$

2.19 (a) Using Equations (2.26) to (2.28)

$$2^{\ell} = L \geq \frac{1}{2p} \text{ levels.}$$ Given that $p = 0.02$,

then $$\ell = \left\lceil \log_2 \frac{1}{0.04} \right\rceil = \left\lceil \log_2 25 \right\rceil$$

Thus, there must be at least 25 quantization levels, or 5 bits per sample, to meet the fidelity criterion. The data rate is:

8000 samples/s × 5 bits/sample = 40,000 bits/s

This data rate needs to be sent in a 4000 Hz bandwidth. Hence, the required bandwidth efficiency is:

$$\frac{R}{W} = \frac{40{,}000 \text{ bits/s}}{4000 \text{ Hz}} = 10 \text{ bits/s/Hz}$$

(b) When the analog signal has a 20 kHz bandwidth, the Nyquist sampling rate is 40 ksamples/s, and the bit rate is:

40,000 samples/s × 5 bits/sample = 200,000 bits/s

Hence the required bandwidth efficiency is:

$$\frac{R}{W} = \frac{200{,}000}{4{,}000} = 50 \text{ bits/s/Hz},$$

which is a challenging requirement!

Chapter 3

3.1 (a) $f_1 = f_2$ and $\phi_1 = \phi_2$

$$\int_{-1.5T_2}^{1.5T_2} s_1(t)s_2(t)\,dt = \int_{-1.5T_2}^{1.5T_2} s_1^2(t)\,dt \neq 0$$

$\therefore$ not orthogonal

(b) $f_1 = \frac{1}{3}f_2$ and $\phi_1 = \phi_2$

Let $\phi_1 = \phi_2 = 0$

$$\int_{-1.5T_2}^{1.5T_2} s_1(t)s_2(t)\,dt = \frac{1}{2}\int_{-1.5T_2}^{1.5T_2} \cos 2\pi\left(\frac{2}{3}f_2\right)t\,dt$$

$$+ \frac{1}{2}\int_{-1.5T_2}^{1.5T_2} \cos 2\pi\left(\frac{4}{3}f_2\right)t\,dt$$

$$= \frac{1}{2}\left[\frac{\sin \frac{4}{3}\pi \frac{t}{T_2}}{\frac{4}{3}\pi\left(\frac{1}{T_2}\right)}\right]_{-1.5T_2}^{1.5T_2} + \frac{1}{2}\left[\frac{\sin \frac{8}{3}\pi \frac{t}{T_2}}{\frac{8}{3}\pi\left(\frac{1}{T_2}\right)}\right]_{-1.5T_2}^{1.5T_2}$$

$$= \frac{\sin 2\pi}{\frac{4}{3}\pi\left(\frac{1}{T_2}\right)} + \frac{\sin 4\pi}{\frac{8}{3}\pi\left(\frac{1}{T_2}\right)} = 0$$

$\therefore$ orthogonal

3.1 (c) $f_1 = 2f_2$ and $\phi_1 = \phi_2$

let $\phi_1 = \phi_2 = 0$

$$\int_{-1.5T_2}^{1.5T_2} s_1(t)\, s_2(t)\, dt = \frac{1}{2}\int_{-1.5T_2}^{1.5T_2} (\cos 2\pi f_2 t + \cos 6\pi f_2 t)\, dt$$

$= 0$ $\quad \therefore$ orthogonal

(d) $f_1 = \pi f_2$ and $\phi_1 = \phi_2$, let $\phi_1 = \phi_2 = 0$

$$\int_a^b s_1(t)\, s_2(t)\, dt = \frac{1}{2}\int_a^b \cos(\pi - 1)\, 2\pi f_2 t\, dt +$$

$$\frac{1}{2}\int_a^b \cos(\pi + 1)\, 2\pi f_2 t\, dt \neq 0$$

$\therefore$ not orthogonal

(e) $f_1 = f_2$ and $\phi_1 = \phi_2 + \pi/2$

$$\int_a^b \sin 2\pi f_2 t \cos 2\pi f_2 t\, dt = 0$$

$\therefore$ orthogonal

(f) $f_1 = f_2$ and $\phi_1 = \phi_2 + \pi$, let $\phi_1 = 0$

$$-\int_a^b \cos^2 (2\pi f_2 t)\, dt \neq 0$$

$\therefore$ not orthogonal

3.2 (a) $\int_{-2}^{2} \psi_1(t)\, \psi_2(t)\, dt = \int_{-2}^{-1} (-A)(-A)\, dt$

$+ \int_{-1}^{0} (A)(-A)\, dt + \int_{0}^{1} (A)(A)\, dt + \int_{1}^{2} (-A)(A)\, dt$

$= \left[A^2 t\right]_{-2}^{-1} + \left[-A^2 t\right]_{-1}^{0} + \left[A^2 t\right]_{0}^{1} + \left[-A^2 t\right]_{1}^{2}$

$= A^2 - A^2 + A^2 - A^2 = 0$

$\int_{-2}^{2} \psi_1(t)\, \psi_3(t)\, dt = \int_{-2}^{-1} (-A)(-A)\, dt + \int_{-1}^{0} (A)(-A)\, dt$

$+ \int_{0}^{1} (A)(-A)\, dt + \int_{1}^{2} (-A)(-A)\, dt = A^2 - A^2 - A^2 + A^2 = 0$

$\int_{-2}^{2} \psi_2(t)\, \psi_3(t)\, dt = \int_{-2}^{0} (-A)(-A)\, dt + \int_{0}^{2} (A)(-A)\, dt$

$= 2A^2 - 2A^2 = 0$

(b) $\int_{-2}^{2} \psi_3^2(t)\, dt = \int_{-2}^{2} A^2\, dt = \left[A^2 t\right]_{-2}^{2}$

$= 2A^2 + 2A^2 = 4A^2$

To be orthonormal, $4A^2 = 1$

$A^2 = 1/4$

$A = 1/2$

3.2 (c)

$x(t)$

1

0 1 2 t

$$x(t) = \varphi_2(t) - \varphi_3(t)$$

where:

$\varphi_2(t)$

1/2

-1/2 -2 -1 0 1 2 t

$\varphi_3(t)$

-2 -1 0 1 2 t

$-\frac{1}{2}$

3.3

$$\int_{-\infty}^{0} e^{t}(1-Ae^{2t})dt + \int_{0}^{\infty} e^{-t}(1-Ae^{-2t})\,dt = 0$$

$$= \int_{-\infty}^{0}(e^{t}-Ae^{3t})\,dt + \int_{0}^{\infty}(e^{-t}-Ae^{-3t})\,dt = 0$$

$$\left[e^{t} - \frac{Ae^{3t}}{3}\right]_{-\infty}^{0} + \left[-e^{-t} + \frac{Ae^{-3t}}{3}\right]_{0}^{\infty} = 0$$

$$1 - \frac{A}{3} - \left[-1 + \frac{A}{3}\right] = 2 - \frac{2A}{3} = 0$$

$$\frac{2A}{3} = 2$$

$$A = 3$$

3.4 Using Equation (2.41)

$$P_B = Q\left(\frac{a_1 - a_2}{2\sigma_0}\right) = Q\left(\frac{1-(-1)}{2}\right)$$

$$= Q(1)$$

Using table B.1, we solve for P_B

$$P_B = 0.1587$$

3.5 Using Equation (2.67)

$$P_B = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$ where $E_b = A^2 T$ for bipolar signaling, and $A = 1$. Thus, $E_b = T$.

$$P_B = Q(x) \leq 10^{-3}$$

$$x = \sqrt{\frac{2E_b}{N_0}} = 3.09 \text{ from Table B.1}$$

$$\frac{E_b}{N_0} = 4.77 \; ; \; \frac{N_0}{2} \text{ is given as } 10^{-3}$$

$$E_b = T = 4.77 \times 10^{-3} \times 2$$

Thus, $R = \frac{1}{T} \leq 104.8$ bits/s.

3.6 (a) $P(s_1) = P(s_2) = 0.5$

Using Equation (2.30)

$$\frac{p(z|s_1)}{p(z|s_2)} \underset{H_2}{\overset{H_1}{\gtrless}} \frac{P(s_2)}{P(s_1)}$$

With equally-likely probabilities, the optimum threshold from Equation (2.31) becomes:

$$z(T) \underset{H_2}{\overset{H_1}{\gtrless}} \frac{a_1 + a_2}{2} = \gamma_0 = \frac{T + (-T)}{2} = 0$$

where $a_1 = \int_0^T dt = T$, and $a_2 = \int_0^T -1\, dt = -T$

(b) $P(s_1) = 0.7$, then $P(s_2) = 0.3$

Using Equation (B.12)

$$\frac{z(a_1 - a_2)}{\sigma_0^2} \underset{H_2}{\overset{H_1}{\gtrless}} \log_e \frac{P(s_2)}{P(s_1)}$$

$$z \underset{H_2}{\overset{H_1}{\gtrless}} \frac{\sigma_0^2}{a_1 - a_2} \log_e \frac{P(s_2)}{P(s_1)} = \gamma_0$$

$$\gamma_0 = \frac{0.1}{2T} \log_e \left(\frac{0.3}{0.7}\right)$$

$$= -\frac{0.04}{T} \text{ volt}$$

3.6 (c) $\gamma_0 = \frac{0.1}{2T} \log_e\left(\frac{0.8}{0.2}\right)$

$= \underline{\frac{0.07}{T}}$ volt

(d) The a priori probabilities have the effect of positioning γ_0 so as to yield a greater probability of correct decisions. For example, when $P(s_1)$ is reduced to 0.2 from 0.5, then γ_0 in Figure 2.25 moves to the right so that samples at the tail of the $p(z|s_2)$ pdf have a greater chance of being declared members of the signal class s_2.

3.7

$$P_E = P(s_1)\int_{-0.2}^{0} \tfrac{1}{2}\,dz + P(s_2)\int_{0}^{0.2} \tfrac{1}{2}\,dz$$

$$= \left[\tfrac{1}{2} z\right]_{-0.2}^{0} = \frac{0.2}{2} = 0.1$$

3.8 (a) 16 levels $= M = 2^k$

$k = 4$ bits/symbol

$$R_s = \frac{R}{\log_2 M} = \frac{10 \text{ Mbits/s}}{4 \text{ bits/symbol}} = 2.5 \text{ Msymbols/s.}$$

$$\text{Min BW} = R_s/2 = 1.25 \text{ MHz}$$

(b) Using Equation (2.76)

$$W = \tfrac{1}{2}(1+r)R_s$$

$$1.375 \text{ MHz} = (1+r)\ 1.25 \text{ MHz}$$

$$r = 0.1$$

3.9 From Equation (2.71)

$$L \geq \frac{1}{2p} = \frac{1}{2 \times 0.001} = 500 \text{ levels}$$

$$\ell = \lceil \log_2 500 \rceil = 9 \text{ bits/sample}$$

$R = 8000$ samples/s $\times$ 9 bits/sample $= 72{,}000$ bits/s.

PCM format with $M = 32$

$$R_s = \frac{R}{\log_2 M} = \frac{72{,}000 \text{ bits/s}}{\log_2 32}$$

$$= 14{,}400 \text{ symbols/s}$$

Theoretical minimum bandwidth without ISI $= R_s/2 = 7200$ Hz.

3.10 (a) $R_s = \dfrac{9600 \text{ bits/s}}{3 \text{ bits/symbol}}$

$= 3200$ symbols/s

(b) $r = \dfrac{W - W_o}{W_o}$ where W_o is the Nyquist minimum bandwidth

$W_o = R_s/2 = 1600$ Hz

$$r = \frac{2400 - 1600}{1600} = 0.5$$

3.11 Voice signal in the frequency range of 300 - 3300 Hz. Sampling is 8000 samples/s.

(a) PAM Transmission

Using Equation (2.76)

$W = \frac{1}{2}(1+r)R_s$; where $R_s = 8000$ pulses/s.

$= \frac{1}{2}(1+1)\,8000$

$= 8$ kHz.

3.11

(b) <u>PCM Transmission</u> - using 8-level quantization.

8000 samples/s × 3 pulses/sample = 24000 pulses/s

$$W = \frac{1}{2}(1+r)R_s = \frac{1}{2}(1+1)\,24000$$

$$= 24 \text{ kHz}.$$

3.11

(c) <u>PCM Transmission</u> - using 128-level quantization.

8000 samples/s × 7 pulses/sample = 56000 pulses/s

$$W = \frac{1}{2}(1+r)R_s = \frac{1}{2}(1+1)\,56{,}000$$

$$= 56 \text{ kHz}.$$

3.12 (a) $W = \frac{1}{2}(1+r)R_s$

$100 \text{ kHz} = \frac{1}{2}(1.6)R_s$

$R = R_s = 125$ ksymbols/s $= 125$ kbits/s

(b) $L = 32 = 2^{\ell} \Rightarrow \ell = 5$

$R = 125$ kbits/s $= 5$ bits/sample $\times f_s$ samples/s

$$f_s = \frac{125 \text{ kbits/s}}{5 \text{ bits/sample}} = 25 \text{ ksamples/s}$$

$$W_{\substack{analog\\max}} = \frac{1}{2} f_s = 12.5 \text{ kHz}$$

3.12 (c) Eight-level PCM Signaling:

$R_s = 125$ ksymbols/s

$R = 125$ ksymbols/s $\times$ 3 bits/symbol $= 375$ kbits/s

$L = 32 = 2^{\ell} \Rightarrow \ell = 5$

$R = 375$ kbits/s $= 5$ bits/sample $\times f_s$ samples/s

$$f_s = \frac{375 \times 10^3}{5} = 75 \text{ kbits/s}$$

$$W_{\substack{analog\\max}} = \frac{1}{2} f_s = 37.5 \text{ kHz}.$$

3.13

Signaling with RZ pulses represents an example of orthogonal signaling. Therefore, for coherent detection, we can use Equation (3.71) as

$$P_B = Q\left(\sqrt{\frac{E_b}{N_0}}\right) = Q\left(\sqrt{\frac{A^2T}{2N_0}}\right)$$

$$10^{-3} = Q\left(\sqrt{\frac{0.01T}{2N_0}}\right) = Q(x)$$

Using Table B.1 to find x, yields x=3.1. Thus,

$$\sqrt{\frac{0.01T}{2\times10^{-8}}} = 3.1, \quad T = 19.2\,\mu s, \quad \text{and} \quad R = 52{,}083\,\text{bits/s}$$

3.14

Signaling with NRZ pulses represents an example of antipodal signaling. Therefore, for coherent detection, we can use Equation (3.70) as

$$P_B = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) = Q\left(\sqrt{\frac{2A^2T}{N_0}}\right)$$

$$10^{-3} = Q\left(\sqrt{\frac{2A^2(1/56{,}000)}{10^{-6}}}\right) = Q(x)$$

Using Table B.1 to find x, yields x=3.1. Thus,

$\sqrt{\frac{2A^2(1/56{,}000)}{10^{-6}}} = 3.1, \quad A^2 = 0.268$. Thus, if there were no signal power loss, the minimum power needed would be approximately 260 mW. With a 3-dB loss, 538 mW are needed.

3.15

The power spectral density for a random bipolar (antipodal) sequence in Equation (1.38) is expressed in the form of

$T_S\left(\dfrac{\sin \pi f T_S}{\pi f T_S}\right)^2$, where T_s is the symbol duration. The total area under the spectral plot is found by integrating as follows:

$$\int_{-\infty}^{\infty} T_S\left(\frac{\sin \pi f T_s}{\pi f T_s}\right)^2 df = 2T_s \int_0^{\infty}\left(\frac{\sin \pi f T_s}{\pi f T_s}\right)^2 df$$

Let $x = \pi f T_S$, then $df = dx/\pi T_S$, and the area is:

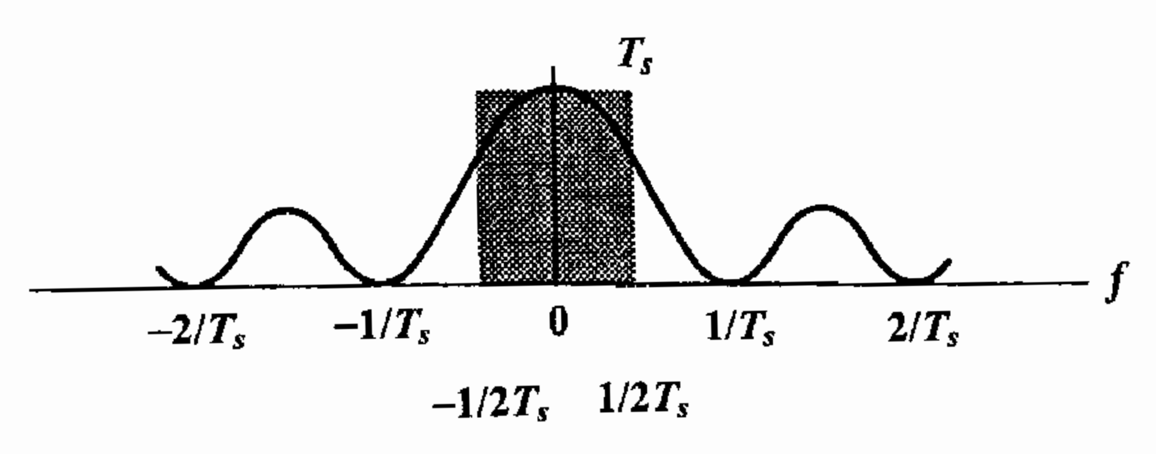
$$\frac{2T_S}{\pi T_S}\int_0^{\infty}\left(\frac{\sin x}{x}\right)^2 dx = \frac{2}{\pi}\frac{\pi}{2} = 1$$

The two-sided Nyquist minimum bandwidth extends from

$$-\frac{1}{2T_S} \text{ to } +\frac{1}{2T_s} = \frac{1}{T_s} = R_s$$

Thus, the one-sided (baseband) bandwidth is $\dfrac{1}{2T_S} = \dfrac{R_S}{2}$.

The sketch below illustrates the rectangular construction having the same area as the signal power spectral density. The width (bandwidth) of this rectangle is R_s (two-sided) and $R_s/2$ (one-sided), which is the same as the Nyquist minimum bandwidth for ideal-shaped bipolar pulses.



3.16

The output of an MF is a time series, such as seen in Figure 3.7b (e.g., a succession of increasing positive and negative correlations to an input sine wave). Such an MF output sequence can be equated to several correlators operating at different starting points of the input time series. Unlike an MF, a correlator only computes an output once per symbol time. A bank of $N = 6$ correlators is shown in Figure 1, where the reference signal for the first one is $s_1(t)$, and the reference for each of the others is a symbol-time-offset copy, $s_1(t-kT)$ of the first reference. It is convenient to refer to the reference signals as *templates*. Since the correlator emulates a matched filter, the "matching" is often provided by choosing each of the $s_i(t)$ templates to be a square-root Nyquist shaped pulse, and thus the overall system transfer function being the product of two root-raised cosine functions, is a raised cosine function. Figure 2 is a pictorial of the 6 shaped-pulse templates, each one occupying 6 symbol times, and each one offset from its staggered neighbor (above and below) by exactly one symbol time. Each of the template signals will be orthogonal to one another, provided that the time offset is chosen to be an integer number of symbols.

Each correlator performs product-integration of the received pulse sequence, $r(t)$, by using its respective template. The time-shifted templates account for the staggered time over which each correlator operates. That is, the first correlator processes the $r(t)$ waveform over the time intervals 0 to 6, then 6 to 12, and so forth. The second correlator operates over the intervals 1 to 7, then 7 to 13, and so forth. The sixth correlator operates over the intervals 5 to 11, then 11 to 17, and so forth. In Figure 1, following the bank of correlators is a commutating switch connecting the correlator outputs to a sampling switch. Startup consists of loading the correlators with 6-symbol durations of the received waveform, after which the commutating switch simply "sits" on the output of each correlator for one symbol duration before moving on to the next correlator. Even though a correlator only produces an output

at the end of a symbol time, the commutating switch acts to form a time-series from the outputs of the staggered correlators. The output of the commutating switch is a discrete approximation of the demodulated raised-cosine (smeared) analog waveform seen in Figure 3.23b. This output is now ready for sampling and detection. The commutating switch itself can be implemented to act as the sampling switch.

Recall that the beneficial attribute of a matched filter or correlator is that it gathers the signal energy that is matched to its template, yielding some peak amplitude at the end of a symbol time. Each correlator, operating on the "smeared" signal, gathers the energy that matches its template over 6-symbol times, and when sampled at the appropriate time, produces an output ready to be detected.

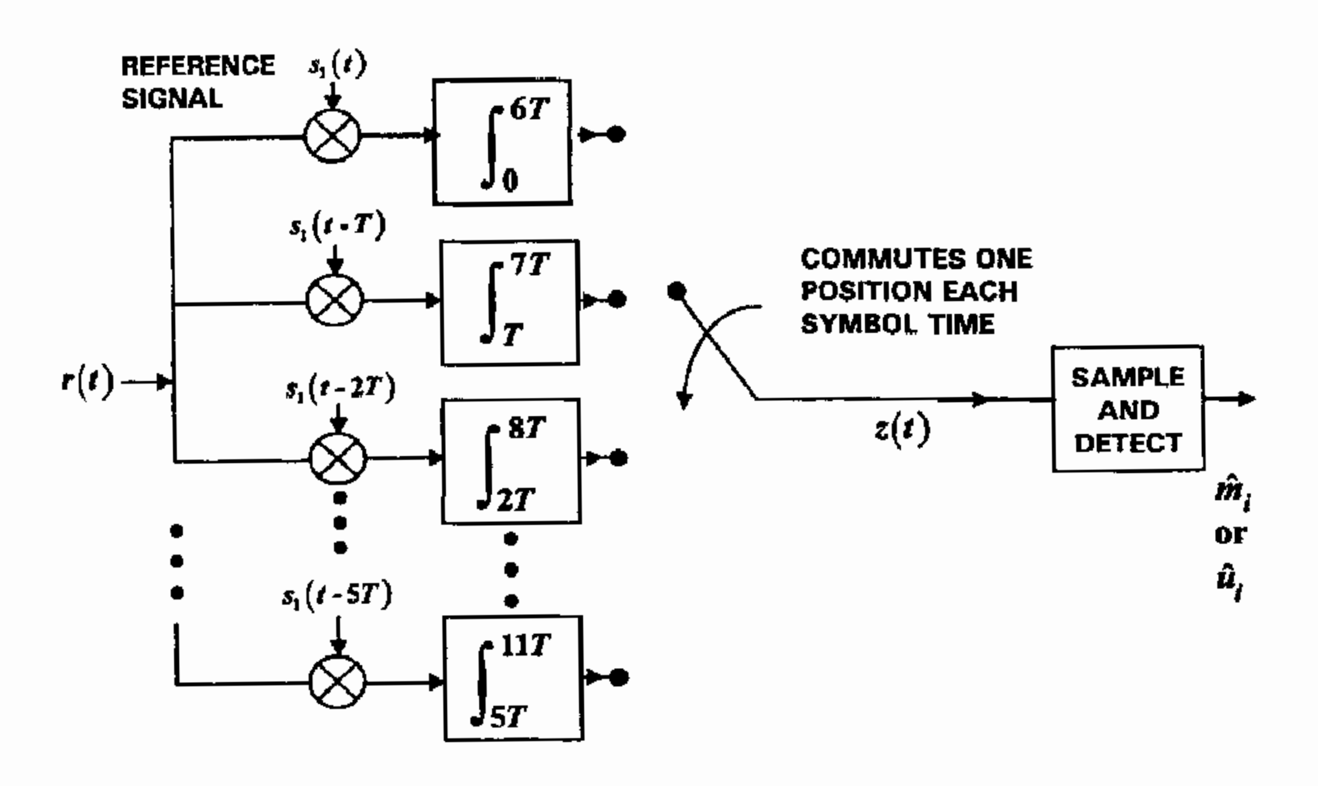


Figure 1

3.16 (cont'd)

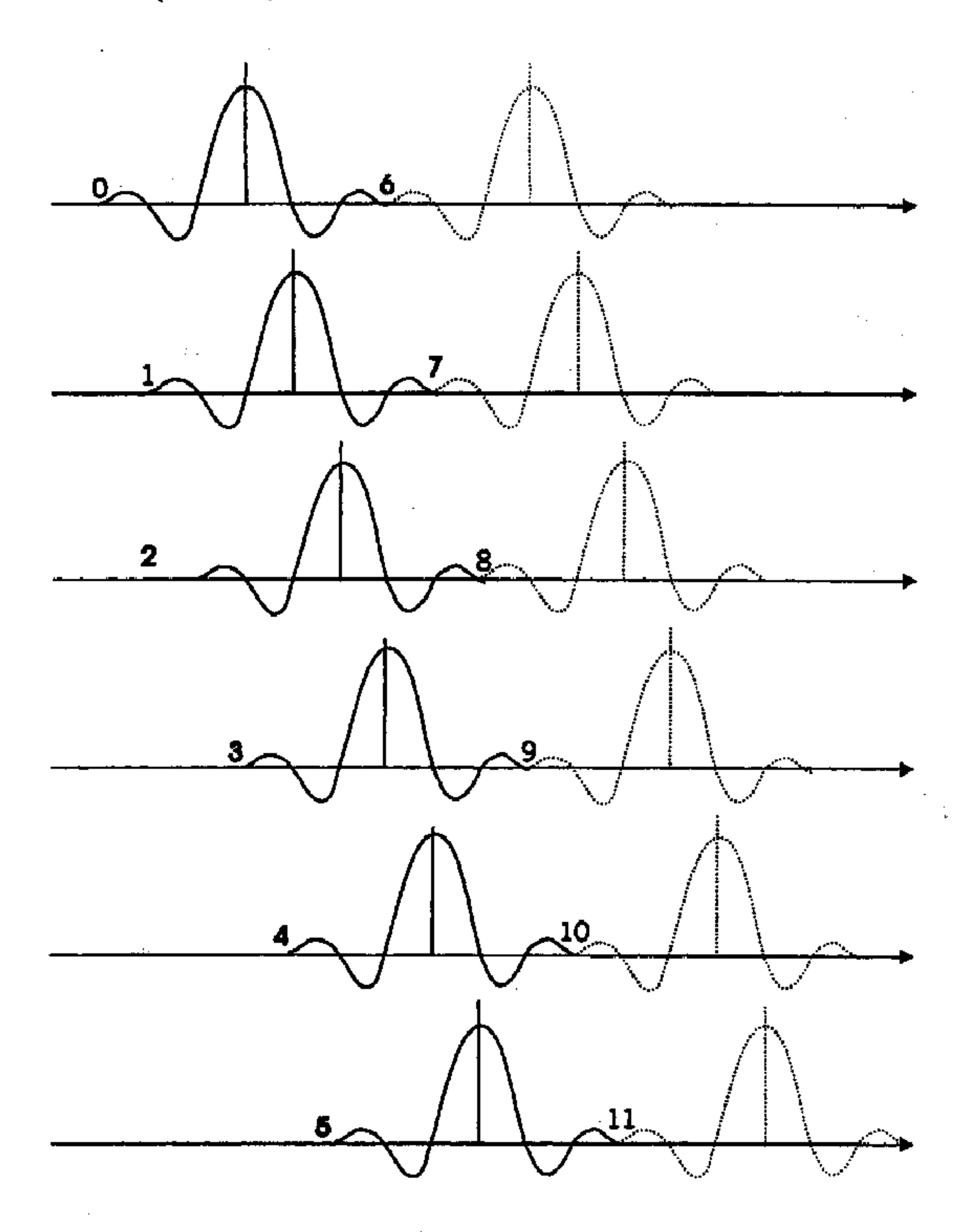


Figure 2

3.16 (cont'd)

For this example, Figure 3 shows the signal into the staggered correlators. We see 6 successive views of the smeared signal appearing as "snapshots" through a sliding window (6-symbol times in duration).

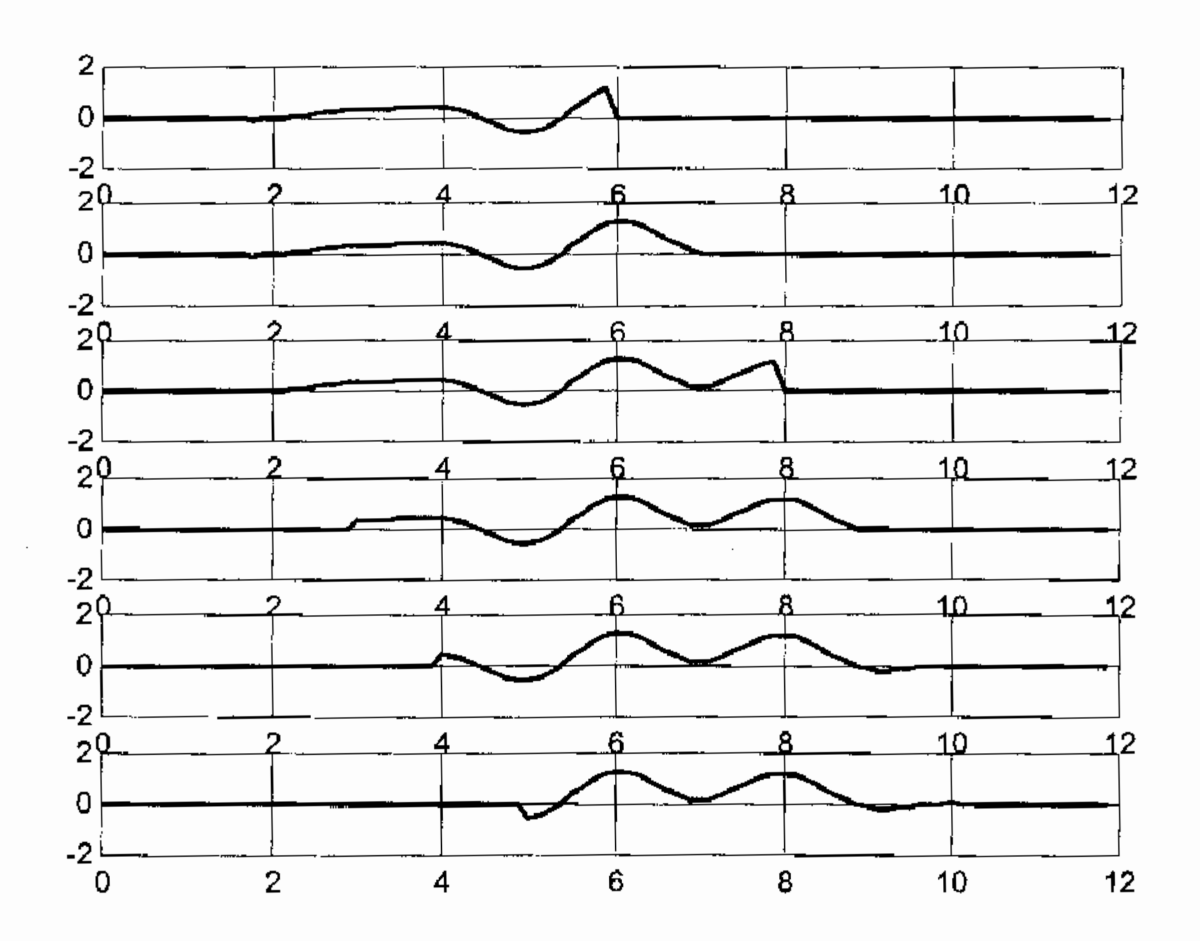


Figure 3. Time intervals processed by successive correlators

3.16 (cont'd)

For this example, Figure 4 shows the output of each successive correlator. We see 6 successive results from each windowed signal in Figure 3 that has here been product-integrated with each of the staggered templates..Note that the signal values at the successive sampling times 6, 7, … , 11 correspond to the PAM signal values that had been sent.

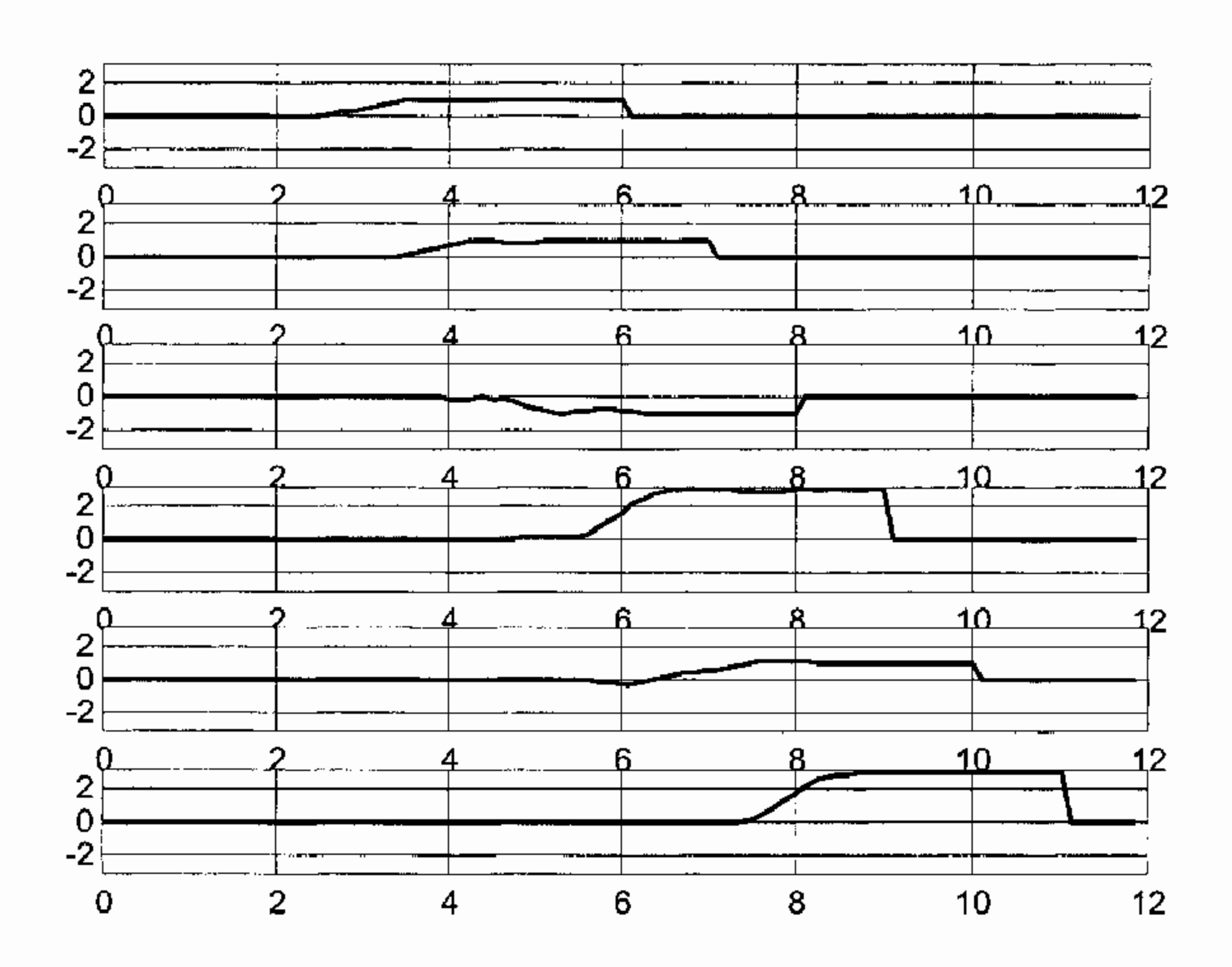


Figure 4. Outputs of successive correlators

3.17

The overall (channel and system) impulse response is $h(t) = \delta(t) + \alpha\delta(t-T)$. We need a compensating (equalizing) filter with impulse response $c(t)$ that forces $h(t)*c(t) = \delta(t)$ and zero everywhere else (zero-forcing filter). The impulse response of the equalizing filter can have the following form:

$$c(t) = c_0\delta(t) + c_1\delta(t-T) + c_2\delta(t-2T) + c_3\delta(t-3T) + \cdots$$

where $\{c_k\}$ are the weights or filter values at times k = 0, 1, 2, 3, ... After equalizing, the system output is obtained by convolving the overall impulse response with the filter impulse response, as follows:

$$h(t)*c(t) = c_0\delta(t) + c_1\delta(t-T) + c_2\delta(t-2T) + c_3\delta(t-3T) + \cdots$$
$$+ \alpha c_0\delta(t-T) + \alpha c_1\delta(t-2T) + \alpha c_2\delta(t-3T) + \cdots$$

We solve for the $\{c_k\}$ weights recursively, forcing the output to be equal to 1 at time $t = 0$, and to be 0 elsewhere.

At t =	Contribution to output	Let $c_0 = 1$	Output
0	c_0	$c_0 = 1$	1
T	$c_1 + \alpha c_0$	$c_1 + \alpha c_0 = 0$ $c_1 = -\alpha c_0$ $c_1 = -\alpha$	0
$2T$	$c_2 + \alpha c_1$	$c_2 + \alpha c_1 = 0$ $c_2 = -\alpha c_1$ $c_2 = +\alpha^2$	0
$3T$	$c_3 + \alpha c_2$	$c_3 + \alpha c_2 = 0$ $c_3 = -\alpha c_2$ $c_3 = -\alpha^3$	0
$4T$	αc_3		$-\alpha^4$

Therefore, the filter impulse response is:

$$c(t) = \delta(t) - \alpha\delta(t-T) + \alpha^2\delta(t-2T) - \alpha^3\delta(t-3T)$$

And the output is:

$$r(t) = h(t) * c(t) = 1\times\delta(t) + 0\times\delta(t-2T) + 0\times\delta(t-3T) - \alpha^4\times\delta(t-4T)$$
$$= \delta(t) - \alpha^4\delta(t-4T)$$

The filter can be designed as a tapped delay line. The longer it is (more taps), the more ISI terms can be forced to zero. If $\alpha = \frac{1}{2}$, then the 4-tap filter described above has an impulse response represented by a 1 plus three 0s, and the resulting ISI has a magnitude of $(1/2)^4 = 1/256$. Further ISI suppression can be accomplished with a longer filter.

3.18 $\{x_{-3} \ldots x_0, \ldots, x_3\} =$

0.1, 0.3, −0.2, 1.0, 0.4, −0.1, 0.1

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} x_0 & x_{-1} & x_{-2} \\ x_1 & x_0 & x_{-1} \\ x_2 & x_1 & x_0 \end{bmatrix} \begin{bmatrix} c_{-1} \\ c_0 \\ c_1 \end{bmatrix}$$

Solution for the weights c_n can best be performed using matrix inversion as expressed by Equation (3.89b). It is helpful to use computer assistance. The result is

$$\begin{bmatrix} c_{-1} \\ c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0.8752 & 0.2593 & -0.2107 \\ -0.3079 & 0.8347 & 0.2593 \\ 0.2107 & -0.3079 & 0.8752 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

3.18 (cont'd.)

Thus, the equalizer weights are

$c_{-1} = 0.2593, \quad c_0 = 0.8347, \quad c_1 = -0.3079$

The output samples $\{z(k)\}$ are found by convolving the input samples and the filter tap weights using Equation (3.86). For the times $k = 0, \pm 1, \dots, \pm 3$, we obtain the equalized sample points

$$\{z(k)\} = 0.1613,\ 0.1678,\ 0.0000,$$
$$1.0000,\ 0.0000,\ -0.1807,\ 0.1143$$

Largest sample magnitude contributing to ISI $= 0.1807$

Sum of ISI magnitudes $= 0.6241$

3.19

Channel response: [0.01 0.02 -0.03 0.10 1.00 0.20 -0.10 0.05 0.02]

Matrix description of problem

$$
\begin{bmatrix}
0.01 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.02 & 0.01 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.03 & 0.02 & 0.01 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.10 & -0.03 & 002 & 0.01 & 0 & 0 & 0 & 0 & 0 \\
1.00 & 0.10 & -0.03 & 0.02 & 0.01 & 0 & 0 & 0 & 0 \\
0.20 & 1.00 & 0.10 & -0.03 & 0.02 & 0.01 & 0 & 0 & 0 \\
-0.10 & 0.20 & 1.00 & 0.10 & -0.03 & 0.02 & 0.01 & 0 & 0 \\
0.05 & -0.10 & 0.20 & 1.00 & 0.10 & -0.03 & 0.02 & 0.01 & 0 \\
0.02 & 0.05 & -0.10 & 0.20 & 1.00 & 0.10 & -0.03 & 0.02 & 0.01 \\
0 & 0.02 & 0.05 & -0.10 & 0.20 & 1.00 & 0.10 & -0.03 & 0.02 \\
0 & 0 & 0.02 & 0.05 & -0.10 & 0.20 & 1.00 & 0.10 & -0.03 \\
0 & 0 & 0 & 0.02 & 0.05 & -0.10 & 0.20 & 1.00 & 0.10 \\
0 & 0 & 0 & 0 & .0.02 & 0.05 & -0.10 & 0.20 & 1.00 \\
0 & 0 & 0 & 0 & 0 & 0.02 & 0.05 & -0.10 & 0.20 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.02 & 0.05 & -0.10 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.02 & 0.05 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.02
\end{bmatrix}
\begin{bmatrix}
C0 \\ C1 \\ C2 \\ C3 \\ C4 \\ C5 \\ C6 \\ C7 \\ C8
\end{bmatrix}
=
\begin{bmatrix}
0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0
\end{bmatrix}
$$

Equivalent form $\mathbf{xc} = \mathbf{z}$

Form of Solution $\mathbf{c} = \left(\mathbf{x}^T\mathbf{x}\right)^{-1}\mathbf{x}^T\mathbf{z}$

Solution

$$
\begin{bmatrix}
C0 \\ C1 \\ C2 \\ C3 \\ C4 \\ C5 \\ C6 \\ C7 \\ C8
\end{bmatrix}
=
\begin{bmatrix}
0.0038 \\ -0.0270 \\ 0.0527 \\ -0.1232 \\ 1.0521 \\ -0.2225 \\ 0.0668 \\ -0.0660 \\ 0.0039
\end{bmatrix}
$$

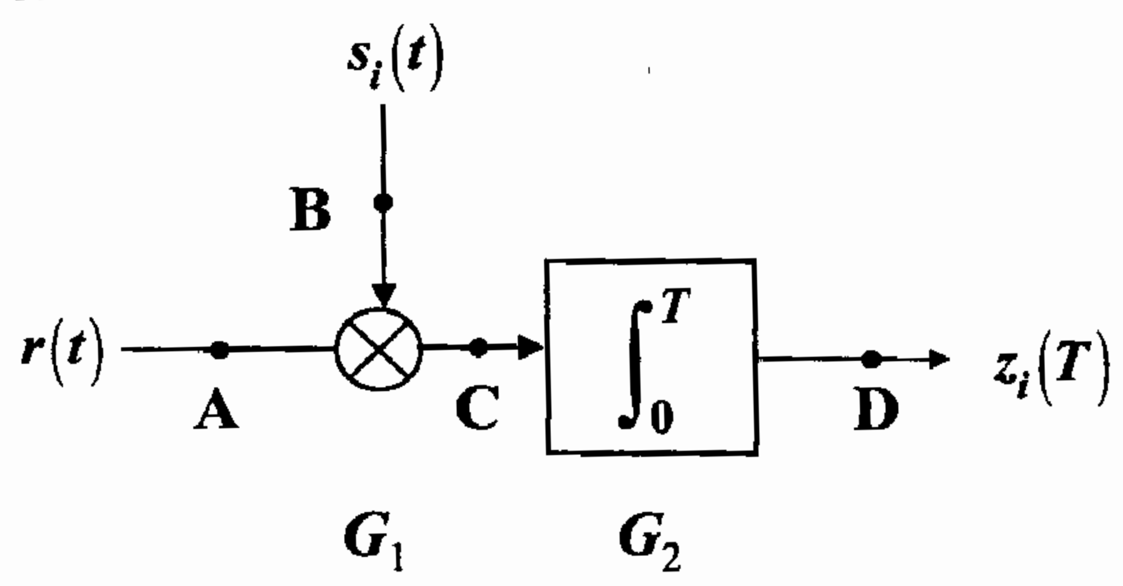
Output of equalized channel:

−0.0000	−0.0003	0.0001	0.0003	−0.0000
0.0000	−0.0000	0.0024	0.9953	0.0111
−0.0947	0.0202	−0.0061	0.0063	−0.0024
−0.0011	0.0001			

Prior to equalization, the maximum single ISI magnitude was 0.2, and the sum of all the ISI magnitude contributions was 0.530.

After equalization, the maximum single ISI magnitude is 0.0947 and the sum of all the ISI magnitude contributions is 0.1450.

3.20



Signals at points A, B, C, and D have units of volts (which characterizes most any signal-processing device). If the transfer function or gain of the multiplier is G_1, then its unit are:

$$r(t) \times s_i(t) \times G_1 = \text{volts (point C)}$$

$$\text{Thus, volts} \times \text{volts} \times G_1 = \text{volts}$$

$$\text{Units of } G_1 = 1/\text{volts}$$

If the gain of the integrator is G_2, then its units are:

$$\text{volts (point C) integrated over } T \text{ seconds} \times G_2 = \text{volts (point D)}$$

$$\text{Thus, volt-seconds} \times G_2 = \text{volts}$$

$$\text{Units of } G_2 = 1/\text{seconds}$$

Therefore, the overall gain or transfer function of the product integrator is 1/volt-seconds. We thus can view the overall transformation as an input energy (volt-squared seconds) times a gain factor of 1/volt-seconds yielding volts/volt-squared-seconds (i.e., an output voltage proportional to energy).

4.1 $P_B = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) = Q\left(\sqrt{\frac{A^2T}{N_0}}\right)$

where $Q(x) \cong \frac{1}{x\sqrt{2\pi}} \exp\left(-x^2/2\right)$ for $x>3$

$A = 1$ mV, $T = \frac{1}{5000}$ s., $N_0 = 10^{-11}$ W/Hz

$$P_B = Q\left(\sqrt{\frac{10^{-6}}{5000\times 10^{-11}}}\right) = Q\left(\sqrt{20}\right) = Q(4.47)$$

$$P_B = \frac{1}{\sqrt{40\pi}} e^{-10} = 4.05 \times 10^{-6}$$

Average no. of errors in one day = 5000 bits/s × 86,400 s/day × 4.05×10^{-6}
$\cong$ 1750 bits in error

4.2 (a) Total bits detected in one day = 1000 bits/s × 86,400 s/day = 8.64×10^{7}

$$P_B = \frac{100}{8.64\times 10^{7}} = 1.16 \times 10^{-6}$$

(b) $P_B = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) = Q\left(\sqrt{\frac{2ST}{10^{-10}}}\right)$

where $S = 10^{-6}$ W and $T = \frac{1}{1000}$ s.

$$P_B = Q\left(\sqrt{\frac{2\times 10^{-6}}{1000\times 10^{-10}}}\right) = Q\left(\sqrt{20}\right) = Q(4.47)$$

$P_B = 4.05 \times 10^{-6}$

No

4.3 Noncoherent BFSK: $E_b/N_0 = 13\text{ dB} = 19.95$

$$P_B = \frac{1}{2}\exp\left(-E_b/2N_0\right) = \frac{1}{2}\exp\left(-19.95/2\right)$$

$$P_B = 2.32 \times 10^{-5}$$

Coherent BPSK: $P_B = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$

$E_b/N_0 = 8\text{ dB} = 6.31$

$$P_B = Q\left(\sqrt{2 \times 6.31}\right) = Q(3.55)$$

$$\cong \frac{1}{\sqrt{2\pi \times 12.6}}\exp\left(-12.6/2\right) = 2.07 \times 10^{-4}$$

∴ Select <u>noncoherent</u> <u>BFSK</u>

4.4 Using Equation (4.43) with the starting bit "1"

1 001100 101 001100 11111 0101
(start)

with the starting bit "0"

0 11 0011 010 11 00 11 00000 1010

Using Equation (4.44) with the starting bit "1"

1 1 0 0 1 1 0 0 0 0 0 1 1 0 0 1 1 0 1 0 1 1 1 1 1
start

with the starting bit "0"

0 0 1 1 0 0 1 1 1 1 1 0 0 1 1 0 0 1 0 1 0 0 0 0 0
start

4.5 (a) Minimum tone separation $= \frac{1}{T}$

$$\Delta f = \frac{1}{T} = \frac{1}{10^{-3}} = 1000 \text{ Hz}$$

Signaling tones are: 1 MHz and 999 kHz

Minimum Bandwidth $= \frac{1}{T} + \frac{2}{T}$

$= 1 \text{ kHz} + 2 \text{ kHz}$

$= 3 \text{ kHz}$

f

$\frac{1}{T}$ $\frac{1}{T}$ $\frac{1}{T}$

(b) Minimum bandwidth for noncoherent MFSK $= \frac{M-1}{T} + \frac{2}{T}$

$$= (M+1)\left(\frac{1}{T}\right) = M+1 \text{ kHz}$$

4.6

$$\sqrt{\frac{2}{T}}\cos(\omega_0 t+\phi)$$

$$A_i(t) = \sqrt{\frac{2E_b}{T}}\cos\omega_0 t \longrightarrow \otimes \longrightarrow \int_0^T \longrightarrow$$

OUTPUT OF CORRELATOR AT t=T

$$z(T) = a_i(T) + n_0(T)$$

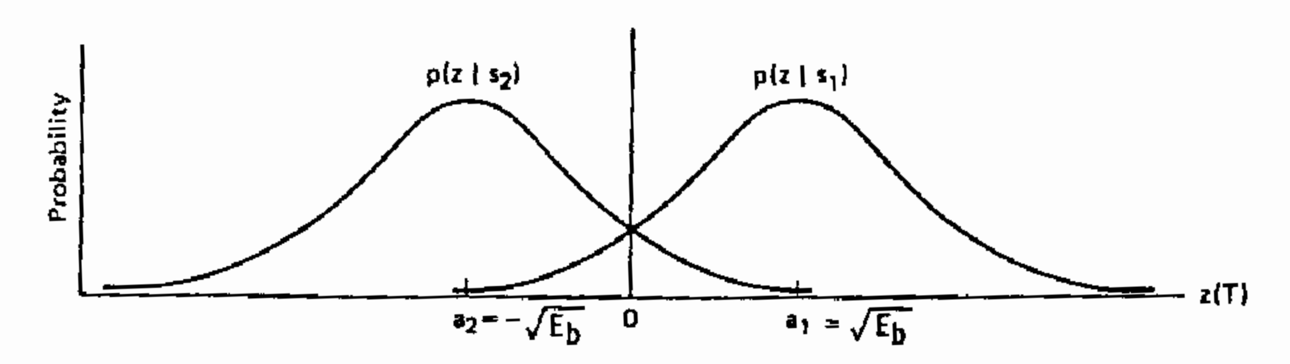
$$a_1(T) = \frac{2}{T}\sqrt{E_b}\int_0^T \cos\omega_0 t \, \cos(\omega_0 t+\phi)\,dt$$

$$= \frac{2}{T}\sqrt{E_b}\int_0^T \frac{1}{2}\left[\cos\phi + \cos(2\omega_0 t+\phi)\right]dt$$

$$= \sqrt{E_b}\cos\phi$$

Similarly, $a_2(T) = -\sqrt{E_b}\cos\phi$

WHEN $\phi = 0$: THE CONDITIONAL pdfs FOR A TYPICAL BINARY RECEIVER ARE



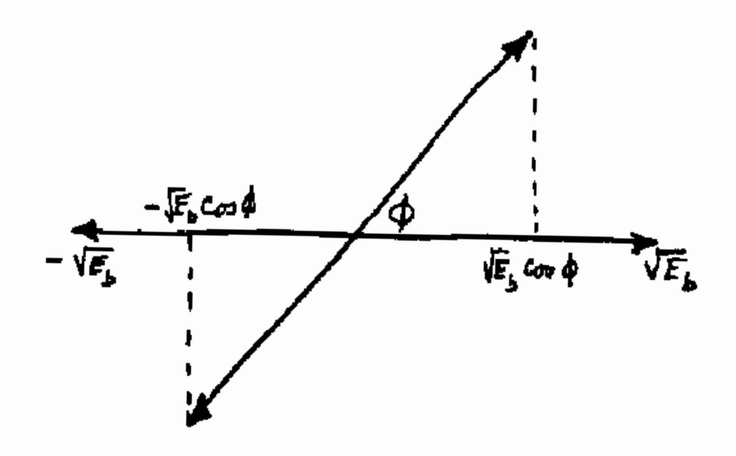
WHERE a_1 AND a_2 REPRESENT THE SIGNAL RESPONSES OF THE MATCHED FILTER, WHEN THERE IS NO PHASE ERROR

$$P_B = Q\left(\frac{a_1 - a_2}{2\sigma_0}\right) \quad \text{WHERE } \sigma_0^2 = \frac{N_0}{2} \quad \text{(SEE APPENDICES B AND C)}$$

$$= Q\left(\frac{\sqrt{E_b} + \sqrt{E_b}}{2\sqrt{N_0/2}}\right) = Q\left(\frac{2\sqrt{E_b}}{2\sqrt{N_0/2}}\right) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

WHEN ϕ IS NONZERO

THE SIGNAL RESPONSES AT THE OUTPUT OF THE MATCHED FILTER BECOME $a_1 = \sqrt{E_b} \cos\phi$ AND $a_2 = -\sqrt{E_b} \cos\phi$



(A) WHEN THERE IS SOME PHASE ERROR ϕ AT THE RECEIVER, THE SIGNAL RESPONSES OF THE MATCHED FILTER BECOME $a_1 \cos\phi$ AND $a_2 \cos\phi$

THEN, $P_B = Q\left(\sqrt{2E_b/N_0}\ \cos\phi\right)$

FOR E_b/N_0 = 9.6 dB = 9.12, AND $\cos 25° = 0.9063$

$$P_B = Q\left(\sqrt{18.24} \times 0.9063\right) = Q(3.87)$$

SINCE $X > 3$ IN $Q(X)$, WE CAN USE THE APPROXIMATION

$$P_B = \frac{1}{X\sqrt{2\pi}} \exp\left(-\frac{X^2}{2}\right) = \frac{1}{3.87\sqrt{2\pi}} \exp\left[-\frac{(3.87)^2}{2}\right] = 5.8 \times 10^{-5}$$

(B) HOW LARGE IS ϕ FOR $P_b = 10^{-3}$?

$$P_B = 10^{-3} = \frac{1}{X\sqrt{2\pi}} \exp\left(-\frac{X^2}{2}\right); \quad X = 3.11525$$

$$\sqrt{2E_b/N_0}\ \cos\phi = 3.11525$$

$$\cos\phi = 3.11525/\sqrt{18.24} = 0.729567; \quad \phi \cong 43°$$

4.7 $E_b = ST = \frac{(0.5)^2}{2}(0.01)$

$= 0.00125$ Joule

$$\rho = \frac{1}{E_b}\int_0^T s_1(t)\, s_2(t)\, dt$$

$$= \frac{1}{E_b}\int_0^T 0.5\cos(2\pi\, 1000t)\; 0.5\cos(2\pi\, 1010t)\, dt$$

$$= \frac{0.25}{0.00125}\int_0^{0.01} \frac{1}{2}\left[\cos 2\pi\, 10t + \cos 2\pi\, 2010t\right] dt$$

$$= 100\left[\frac{\sin 2\pi\, 10t}{20\pi} + \frac{\sin 2\pi\, 2010t}{4020\pi}\right]_0^{0.01}$$

$$\rho = [0.935 + 0.005] = 0.94$$

$$P_B = Q\left(\sqrt{\frac{E_b(1-\rho)}{N_0}}\right) = Q\left(\sqrt{\frac{0.00125\,(0.06)}{0.0002}}\right)$$

$$= Q(0.612)$$

$$P_B = 0.27$$

The error is much greater than if the tone spacing required for coherent <u>orthogonal</u> signaling, $\frac{1}{2T} = 50$ Hz had been used, instead of the 10 Hz specified.

4.8 $s_1(t) = \sqrt{2E/T} \cos \omega_0 t$

$s_2(t) = \sqrt{E/2T} \cos(\omega_0 t + \pi)$

From Equations (4.21) to (4.23), we can characterize these waveforms using the basis function, $\psi_1(t) = \sqrt{\frac{2}{T}} \cos \omega_0 t$.

$s_1(t) = \sqrt{E} \ \psi_1(t)$

$s_2(t) = \frac{1}{2}\sqrt{E}\ \psi_1(t)$

$$\gamma_0 = \frac{a_1 + a_2}{2} = \frac{\sqrt{E} + \left(-\frac{1}{2}\sqrt{E}\right)}{2} = \frac{\sqrt{E}}{4}$$

$$\gamma_0 = \sqrt{E/2}$$

4.9 (a) $P_B = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$

$E_b/N_0 = 6.8 \text{ dB} = 4.786$

$P_B = Q\left(\sqrt{2 \times 4.786}\right) = Q(3.09)$

From Table B.1, $P_B = 10^{-3}$

4.9 (b)

$p(z|A_2)$ $p(z|A_1)$

$-\sqrt{E}$ 0 $\sqrt{E}$ $z(T)$

$\gamma = 0.1\sqrt{E}$

$$P_B = \frac{1}{2}\int_{-\infty}^{0.1\sqrt{E}} p(z|A_1)\,dz + \frac{1}{2}\int_{0.1\sqrt{E}}^{\infty} p(z|A_2)\,dz$$

$$= \frac{1}{2}\int_{-\infty}^{0.1\sqrt{E}} \frac{1}{\sigma_0\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{z-\sqrt{E}}{\sigma_0}\right)^2\right] dz$$

$$+ \frac{1}{2}\int_{0.1\sqrt{E}}^{\infty} \frac{1}{\sigma_0\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{z+\sqrt{E}}{\sigma_0}\right)^2\right] dz$$

Let $u_1 = \dfrac{z-\sqrt{E}}{\sigma_0}$; $\sigma_0\, du_1 = dz$

let $u_2 = \dfrac{z+\sqrt{E}}{\sigma_0}$; $\sigma_0\, du_2 = dz$

$$P_B = \frac{1}{2}\int_{-\infty}^{-\frac{0.9\sqrt{E}}{\sigma_0}} \frac{1}{\sqrt{2\pi}} e^{-\frac{u_1^2}{2}} du_1 + \frac{1}{2}\int_{\frac{1.1\sqrt{E}}{\sigma_0}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{u_2^2}{2}} du_2$$

For the symmetrical Gaussian function,

$$\int_{-\infty}^{-x} e^{-\frac{u^2}{2}} du = \int_{x}^{\infty} e^{-\frac{u^2}{2}} du$$

$$P_B = \frac{1}{2} Q\left(\frac{0.9\sqrt{E}}{\sigma_0}\right) + \frac{1}{2} Q\left(\frac{1.1\sqrt{E}}{\sigma_0}\right)$$

For binary matched filter detection, we can write $E = E_b$ and $\sigma_0^2 = N_0/2$.

$$P_B = \frac{1}{2} Q\left(0.9\sqrt{\frac{2E_b}{N_0}}\right) + \frac{1}{2} Q\left(1.1\sqrt{\frac{2E_b}{N_0}}\right)$$

$E_b/N_0 = 6.8 \text{ dB} = 4.786$

$$P_B = \frac{1}{2} Q\left(0.9 \times 3.09\right) + \frac{1}{2} Q\left(1.1 \times 3.09\right)$$

$$P_B = \frac{1}{2} Q\left(2.78\right) + \frac{1}{2} Q\left(3.4\right)$$

$$= \frac{1}{2}\left(0.0027\right) + \frac{1}{2}\left(0.0003\right)$$

$$= 1.4 \times 10^{-3}$$

4.9 (c) From Equations (B.8) to (B.12)

$$\frac{p(z|s_1)}{p(z|s_2)} \underset{H_2}{\overset{H_1}{\gtrless}} \frac{P(s_2)}{P(s_1)}$$

$$\frac{z(a_1 - a_2)}{\sigma_0^2} - \cancelto{0}{\frac{(a_1^2 - a_2^2)}{2\sigma_0^2}} \underset{H_2}{\overset{H_1}{\gtrless}} \ln\left[\frac{P(s_2)}{P(s_1)}\right]$$

$$\frac{z(2\sqrt{E})}{N_0/2} \underset{H_2}{\overset{H_1}{\gtrless}} \ln\left[\frac{P(s_2)}{P(s_1)}\right]$$

$$z \underset{H_2}{\overset{H_1}{\gtrless}} \frac{N_0/2}{2\sqrt{E}} \ln\left[\frac{P(A_2)}{P(A_1)}\right] = \gamma$$

$$\gamma = 0.1\sqrt{E} = \frac{N_0}{4\sqrt{E}} \ln\left[\frac{P(A_2)}{P(A_1)}\right]$$

$$\ln\left[\frac{P(A_2)}{P(A_1)}\right] = \frac{0.4E}{N_0} = 0.4 \times 4.786$$

$$= 1.914$$

$$\frac{P(A_2)}{1-P(A_2)} = \exp(1.914) = 6.782$$

$$P(A_2) = 6.782 - 6.782\,P(A_2)$$

$$P(A_2) = \frac{6.782}{7.782} = 0.87$$

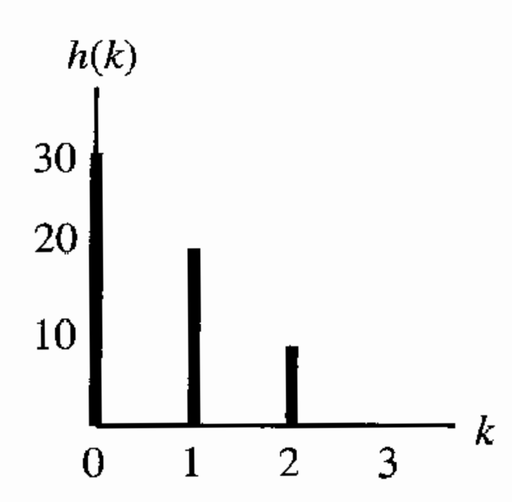
$$P(A_1) = 1 - P(A_2) = 0.13$$

4.10 (a)

A matched filter (MF) is defined as a filter whose impulse response, $h(t)$, is the time-reversed and delayed version of the input signal that the filter is matched to. Thus, for a received signal $s(t)$, where noise is being neglected, the form of $h(t)$ is $h(t) = s(T - t)$, where T is the time duration of the signal waveform being matched. In order for the filter to be realizable, the output must be delayed by the time T that it takes for the entire signal to be received at the input. The discrete signal waveform, $s(k)$, in Figure P4.1, can be described analytically with delta functions as $s(k) = 0\,\delta(0) + 10\,\delta(1) + 20\,\delta(2) + 30\,\delta(3)$. Since the duration of $s(k)$ is three time intervals, we now represent the impulse response of a filter (in discrete form) that is matched to $s(k)$ as

$$h(k) = s(3-k) = 0\,\delta(3-0) + 10\,\delta(3-1) + 20\,\delta(3-2) + 30\,\delta(3-3)$$

$$= 30\,\delta(0) + 20\,\delta(1) + 10\,\delta(2) + 0\,\delta(3)$$



$h(k)$ corresponds to the first of two time-reversals that takes place in the matched-filter detection process. With $s(k)$ at the input to a filter, the ouput, $z(k)$ is obtained from convolving $s(k)$ with the impulse response of the filter. For continuous signals, this takes the form $z(t) = s(t) * h(t) = \int_0^t s(\tau)\, h(t-\tau) d\tau$. When the filter is an MF, so that $h(t) = s(T - t)$, then within the convolution integral

$$h(t-\tau) = s[T-(t-\tau)] = s(T-t+\tau)$$

$$\text{and} \quad z(t) = \int_0^t s(\tau)\, s(T-t+\tau) d\tau$$

During integration, time t is held constant, and we integrate with respect to the dummy variable τ. This convolution integral represents the second time-reversal step in the matched-filter detection process. After sampling $z(t)$ at $t = T$, we have the predetection signal $z(T) = \int_0^T s(\tau)\, s(\tau) d\tau$. Recall that noise is being neglected. The discrete form of this convolution can be written as

$$z(k) = s(k) * h(k) = \sum_{n=0}^{N-1} s(n)\, h(k-n)$$

For the MF in this problem

$$h(k) = s(3-k) = 30\, \delta(0) + 20\, \delta(1) + 10\, \delta(2) + 0\, \delta(3)$$

and within the convolution summation

$$h(k-n) = s[3-(k-n)] = s(3-k+n)$$

$$\text{Therefore,} \quad z(k) = \sum_{n=0}^{N-1} s(n)\, s[n+(3-k)]$$

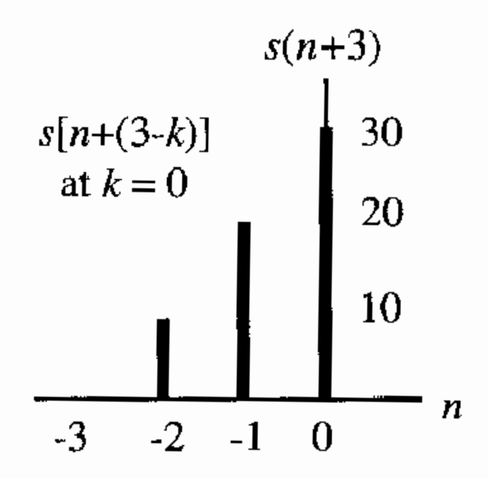
We say again, that this convolution represents the second time-reversal in the MF detection process, so that when we plot $s[n+(3-k)]$ at points $k = 0$ and $k = 1$ for example, we can recognize that this reversal now results in functions $s(n+3)$ and $s(n+2)$ (see below) having a similar appearance to the input $s(k)$.

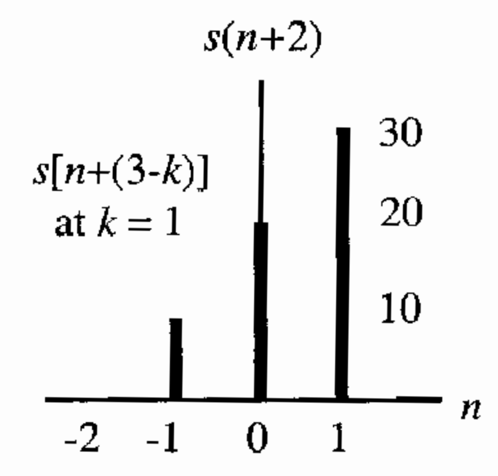
We now compute points for plotting the output, $z(k)$, where k is the input and output time index, and n is a dummy time variable.

$$z(k) = \sum_{n=0}^{N-1} s(n)\, s[n+(3-k)]$$

For $k = 0$: $z(0) = s(0)\, s(3) + s(1)\, s(4) + \cdots = 0$

For $k = 1$: $z(1) = s(0)\, s(2) + s(1)\, s(3) + + s(2)\, s(4) + \cdots = 300$





For $k = 2$: $z(2) = s(0)\, s(1) + s(1)\, s(2) + s(2)\, s(3) + s(3)\, s(4) + \cdots = 800$

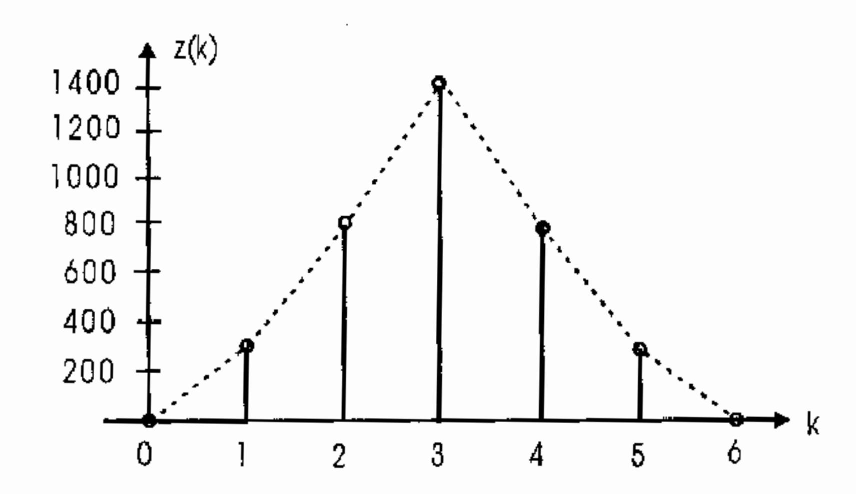
For $k = 3$: $z(3) = s(0)\, s(0) + s(1)\, s(1) + s(2)\, s(2) + s(3)\, s(3) + \cdots = 1400$

For $k = 4$: $z(4) = s(1)\, s(0) + s(2)\, s(1) + s(3)\, s(2) + \cdots = 800$

For $k = 5$: $z(5) = s(2)\, s(0) + s(3)\, s(1) + s(4)\, s(2) + \cdots = 300$

For $k = 6$: $z(6) = s(3)\, s(0) + s(4)\, s(1) + \cdots = 0$

The plot of $z(k)$ versus k is shown below. The maximum output value is 1400.



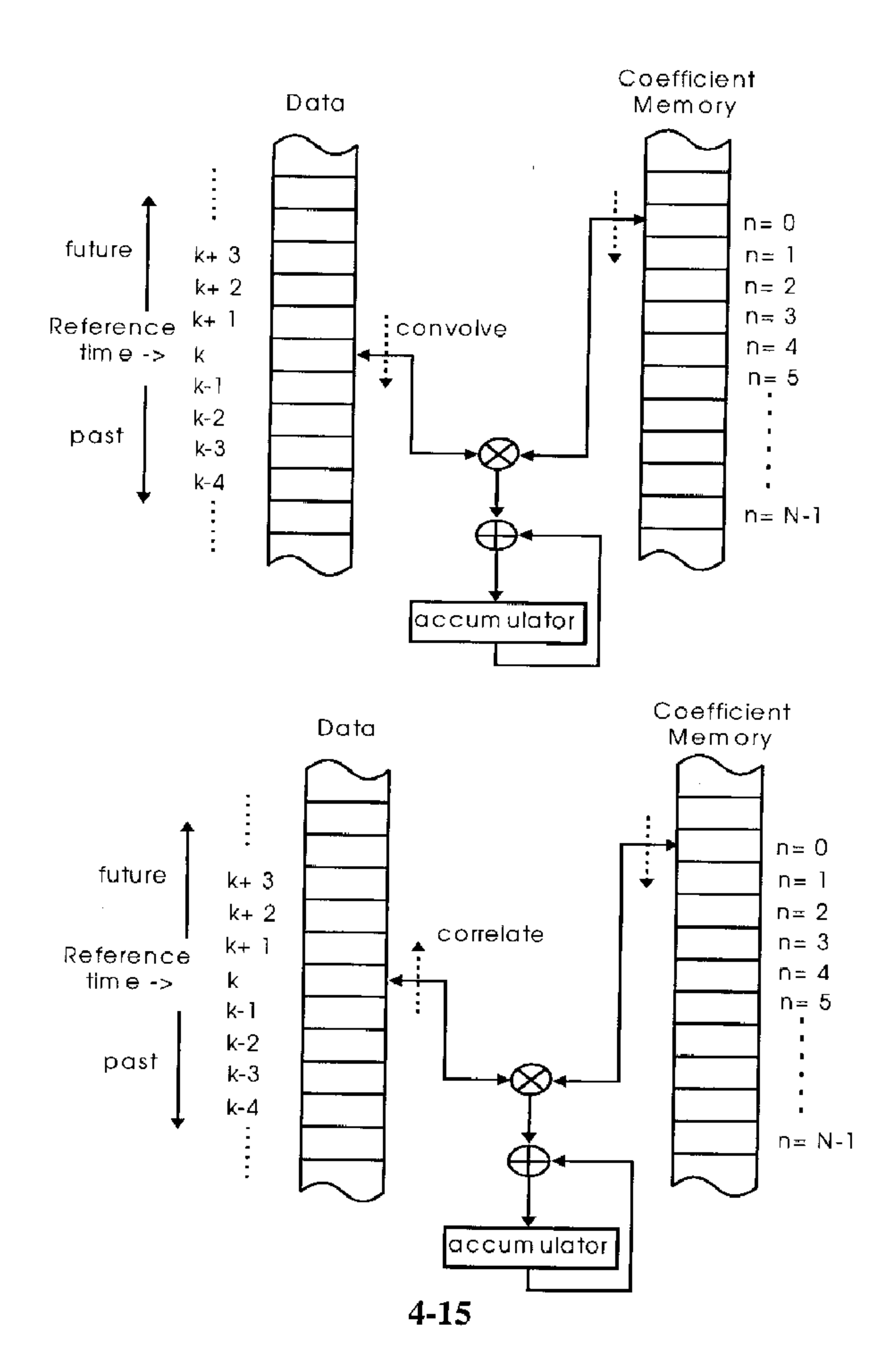
4.10 (b)
A correlator and convolver perform tasks that are remarkably similar, so similar that there is a potential for misunderstanding or for confusing the two functions. The process of convolution and correlation are show below.

Convolution: $$z(k) = \sum_{n=0}^{N-1} s(n)\, h(k-n)$$

Correlation: $$z(k) = \sum_{n=0}^{N-1} s(n) h(k+n)$$

A visualization of the difference between the two functions can be seen by examining the dummy index "*n*" of summation. The index "*n*" points to both the data sample and to the location in coefficient memory containing the coefficient sample required to interact with the selected data sample. A figure emphasizing this mapping is shown below. We can initially locate the data pointer at address "*k*" and then offset the pointer to data samples in the past (*k-n*), and to data samples in the future (*k+n*). When we decrement the address pointer into past samples (relative to address "*k*"), we perform convolution. When we increment the address pointer into future samples (relative to address "*k*"), we perform correlation.

If you accidentally built a circuit that correlated a signal with its time reversed copy, the output would take the form of the convolver $z(k) = \sum_{n=0}^{N-1} s(n)\, h(k-n)$. For this example the input sequence is: $s(k) = 0\,\delta(0) + 10\,\delta(1) + 20\,\delta(2) + 30\,\delta(3)$ and the time-reversed sequence is $s(-k) = 30\,\delta(-3) + 20\,\delta(-2) + 10\,\delta(-1) + 0\,\delta(0)$


Data
Coefficient Memory
future
Reference time ->
past
k+ 3
k+ 2
k+ 1
k
k-1
k-2
k-3
k-4
convolve
n= 0
n= 1
n= 2
n= 3
n= 4
n= 5
n= N-1
accumulator
Data
Coefficient Memory
future
Reference time ->
past
k+ 3
k+ 2
k+ 1
k
k-1
k-2
k-3
k-4
correlate
n= 0
n= 1
n= 2
n= 3
n= 4
n= 5
n= N-1
accumulator

4.10 (b) (cont'd.)

We compute $z(t)$ as follows:

For $k = 0$: $z(0) = s(0)\,s(0) + \cdots = 0$

For $k = 1$: $z(1) = s(1)\,s(0) + s(0)\,s(1) + \cdots = 0$

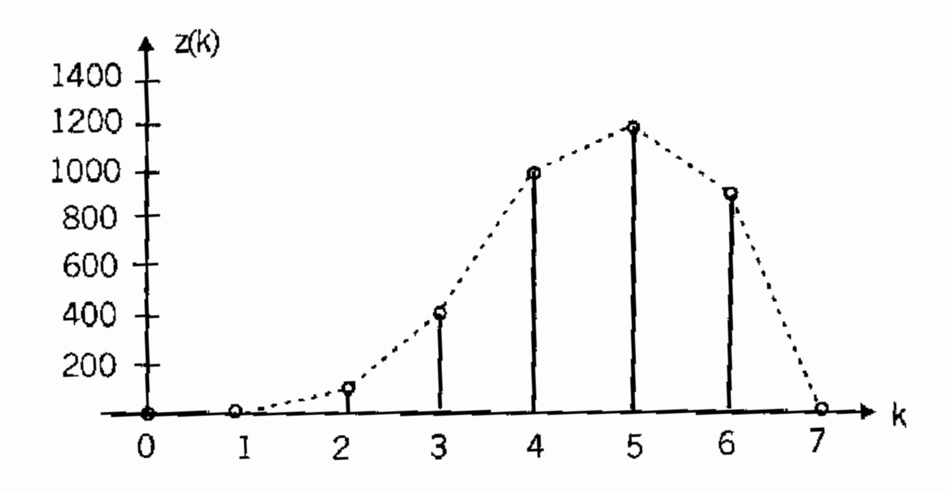
For $k = 2$: $z(2) = s(2)\,s(0) + s(1)\,s(1) + \cdots = 100$

For $k = 3$: $z(3) = s(3)\,s(0) + s(2)\,s(1) + s(1)\,s(2) + \cdots = 400$

For $k = 4$: $z(4) = s(4)\,s(0) + s(3)\,s(1) + s(2)\,s(2) + s(1)\,s(3) + \cdots = 1000$

For $k = 5$: $z(5) = s(5)\,s(0) + s(4)\,s(1) + s(3)\,s(2) + s(2)\,s(3) + \cdots = 1200$

For $k = 6$: $z(6) = s(3)\,s(3) + \cdots = 900$



(c) A valid MF output has symmetry. In part (b) above, we see that the output of a convolver does not have symmetry.

(d) At the peak output, the SNR for the correlator is always greater than that of the convolver. The optimum way of detecting signals in AWGN is with an MF. If the input consist of noise only, the two outputs are identical.

4.10 (cont'd)

Here are some additional thoughts regarding the difference between convolution and correlation. Convolution between two series $x(n)$ and $h(n)$ is described as the running weighted sum formed as follows. One of the sequences, say $x(n)$ is reversed to form a new series $x(-n)$. This new series is then shifted by the amount k, to obtain the sequence $x(k-n)$. This time reversed and time shifted series is multiplied point by point with the second series $h(n)$ forming a new series $x(k-n)h(n)$. The product is summed and the summation represents the value of the convolution of the two series for offset k. The convolution can be formed for positive and negative offsets k.

Correlation between two series $x(n)$ and $h(n)$ is described as the running weighted sum formed as follows. One of the sequences, say $x(n)$ is shifted by the amount k, to obtain the sequence $x(k+n)$. This time shifted series is multiplied point by point with the second series $h(k)$ forming a new series $x(k+n)h(n)$. The product is summed and the summation represents the value of the correlation of the two series for offset k. The correlation can be formed for positive and negative offsets k.

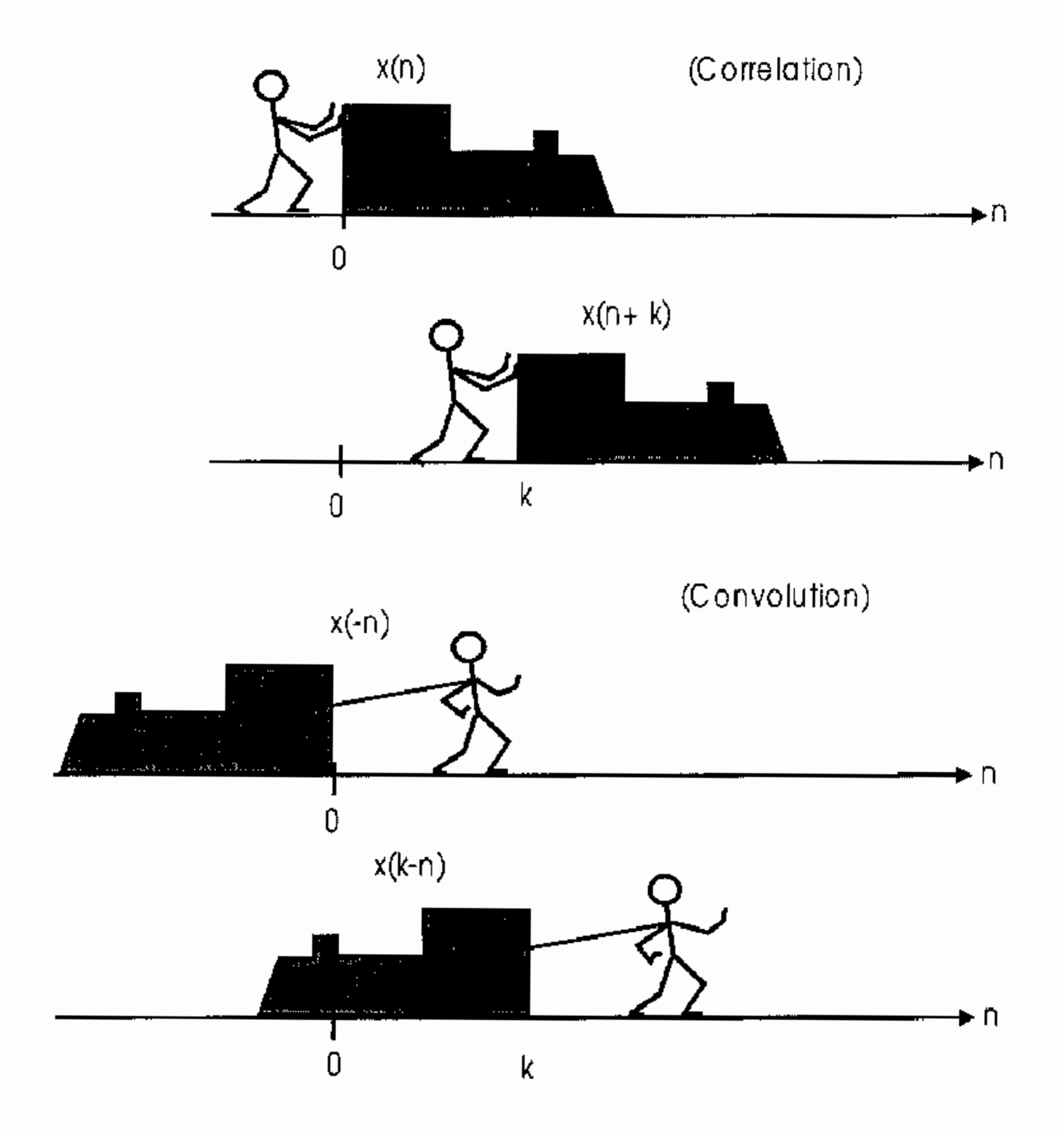
We see that the difference between the two operations convolution and correlation is that for convolution one of the series is time reversed prior to performing the running weighted summation while for correlation the running weighted summation is performed without the time reversal.

For both cases, the sliding sequence can be visualized as being shifted to the left (in the direction of negative time) sufficiently far that there is no overlap of the two series. The shifting series is then moved towards the right to form the running weighted summation as a function of the offset parameter "k".

Traditionally, the convolution sum is represented as a summation from 0-to-N. These limits reflect the constraint that the two series are causal, both starting at index $n = 0$.

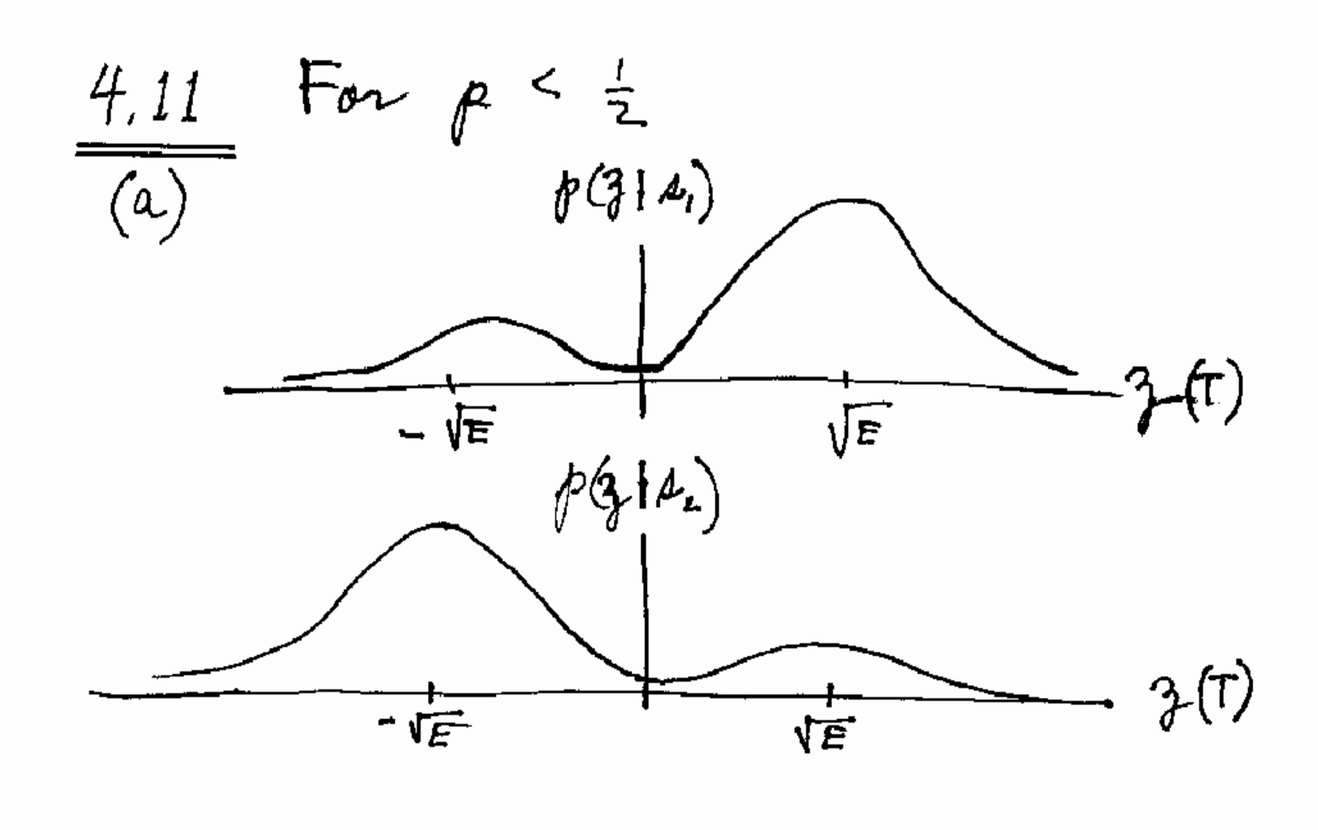
The correlation sum is represented as a summation from $-N$ to $+N$. These limits reflect the constraint that the two series are non-causal, both starting at index $-N/2$ and extending to $+N/2$. Here the time origin is considered arbitrary and is used to indicate the offset for which the product sum of two identical sequences achieve a maximum value.

We can visualize the difference between the two operations, convolution and correlation by assuming both sequences are causal and we can tag the time origin of the sliding series. Imagine a little animated man moving the sliding series past the stationary one. The man *must* stay at the leading edge (initially index zero) as he moves the series. Thus, if the series is time reversed he must pull the series, and if it is not time reversed he must push the series.


x(n)
(Correlation)
n
0
x(n+ k)
n
0
k
(Convolution)
x(-n)
n
0
x(k-n)
n
0
k

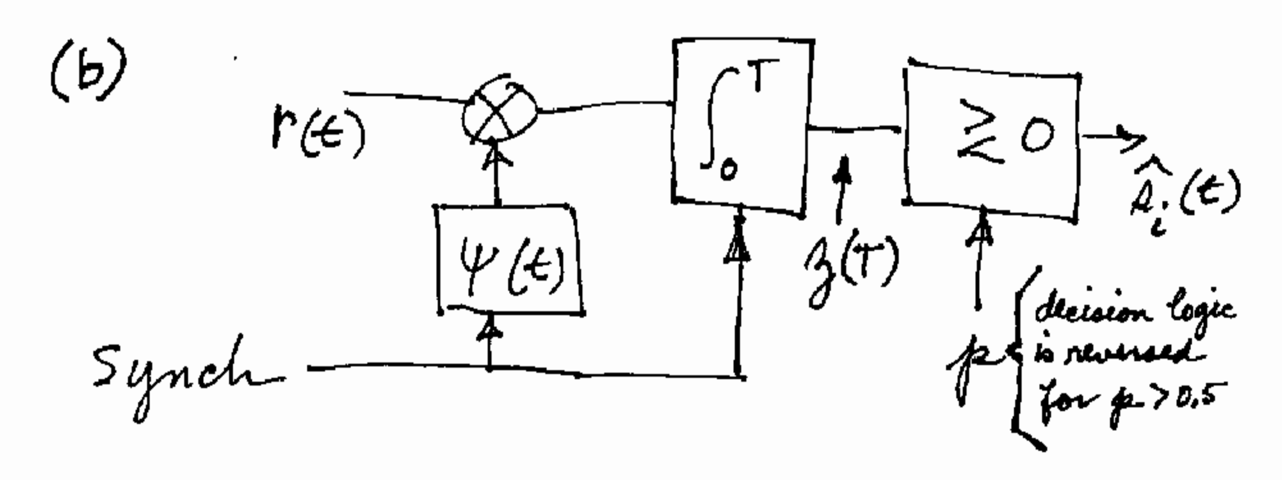
4.11 For $p < \frac{1}{2}$

(a)



$$p(z|s_1) = (p)\, p(z|s_2) + (1-p)\, p(z|s_1)$$
$$p(z|s_2) = (1-p)\, p(z|s_2) + (p)\, p(z|s_1)$$

(b)



4.11 (cont'd.)

(c) With $p = 0.1$ and $\frac{E_b}{N_0} = \infty$, $P_B = 0.1$

With $p = 0$ and $\frac{E_b}{N_0} = 7\,dB$,

For antipodal signals $P_B = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$

$P_B = Q(3.167) = 8 \times 10^{-4}$

$\therefore$ The latter is preferred.

4.12

(a) $P_B \cong \frac{P_E}{k} = \frac{10^{-5}}{4} = 2.5 \times 10^{-6}$

(b) $P_B = \frac{2^{k-1}}{2^k - 1}(P_E) = \frac{2^3}{2^4 - 1} P_E$

$= \frac{8}{15} \times 10^{-5} = 5.3 \times 10^{-6}$

4.13 $P_E = (M-1)\,Q\left(\sqrt{\frac{E_s}{N_0}}\right)$

$E_s = \frac{A^2}{2}T = \frac{(10^{-3})^2}{2} \times 0.2\times 10^{-3} = 10^{-10}$

$P_E = (8-1)\,Q\left(\sqrt{\frac{10^{-10}}{2\times 10^{-11}}}\right) = 7Q(2.236)$

Using Table B.1, $P_E = 7 \times 0.0127$
$= 8.89\times 10^{-2}$

$P_B = \frac{2^{k-1}}{2^k-1}P_E = \frac{2^2}{2^3-1}P_E = \frac{4}{7}P_E$

$= 5\times 10^{-2}$

4.14 (a) With roll-off $r = 1$, and no ISI,

$W_{DSB} = (1+r)R_s$

$50\ kHz = 2R_s$; $R_s = 25k$ symbols/s

$k = \log_2 M = \frac{R}{R_s} = \frac{100\ kbits/s}{25\ ksymbols/s} = 4$

$\therefore M = 16$

Since a Gray code is used, $P_B \approx \frac{P_E}{\log_2 M}$

$P_E = (\log_2 M)P_B = 4\times 10^{-3}$

$P_E = 2Q\left[\left(\sqrt{\frac{2E_s}{N_0}}\right)\sin\left(\frac{\pi}{M}\right)\right] = 4\times 10^{-3}$

$Q(x) = 2\times 10^{-3}$

From Table B.1, $x = 2.88$

$$\left(\sqrt{\frac{2E_s}{N_0}}\right) \sin\left(\frac{\pi}{M}\right) = 2.88$$

$$\sqrt{\frac{2E_s}{N_0}} = \frac{2.88}{\sin\left(\frac{\pi}{16}\right)} = \frac{2.88}{0.19509} = 14.76$$

$$\frac{E_s}{N_0} = 108.9 = 20.4 \text{ dB}$$

(b) $$\frac{E_b}{N_0} = \frac{108.9}{k} = \frac{108.9}{4} = 27.2 = 14.3 \text{ dB}$$

4.15 $$\frac{E_s}{N_0} = k\frac{E_b}{N_0} = 3 \times 10 = 30$$

$$= 14.77 \text{ dB}$$

$$P_E = 2Q\left[\left(\sqrt{\frac{2E_s}{N_0}}\right) \sin\left(\frac{\pi}{\sqrt{2}\, M}\right)\right]$$

$$= 2Q\left[\sqrt{60}\ \sin\left(\frac{\pi}{8\sqrt{2}}\right)\right]$$

$$= 2Q\left[\sqrt{60}\ (0.2741)\right] = 2Q(2.123)$$

Using Table B.1

$$P_E = 2 \times 0.169 = 3.38 \times 10^{-2}$$

4.16 Coherent 8-ary FSK: $\frac{E_b}{N_0} = 8\text{ dB} = 6.31$

$$P_E(M) = (M-1)\, Q\left(\sqrt{E_s/N_0}\right)$$

$$\frac{E_s}{N_0} = k\,\frac{E_b}{N_0} = 3 \times 6.31 = 18.93$$

$$P_E(M) = 7\,Q\left(\sqrt{18.93}\right) = 7Q(4.35)$$

$$\cong \frac{1}{4.35\sqrt{2\pi}} \exp\left[-\frac{(4.35)^2}{2}\right]$$

$$= 4.98 \times 10^{-5}$$

$$P_B = \frac{2^{k-1}}{2^k - 1} P_E = \frac{4}{7} P_E = 2.85 \times 10^{-5}$$

Coherent 8-ary PSK: $\frac{E_b}{N_0} = 13\text{ dB} \cong 20$

$$\frac{E_s}{N_0} = k\,\frac{E_b}{N_0} = 3 \times 20 = 60$$

$$P_E(M) = 2Q\left[\left(\sqrt{\frac{2E_s}{N_0}}\right)\sin\left(\frac{\pi}{M}\right)\right] = 2Q\left[\sqrt{120}\,\sin\left(\frac{\pi}{8}\right)\right]$$

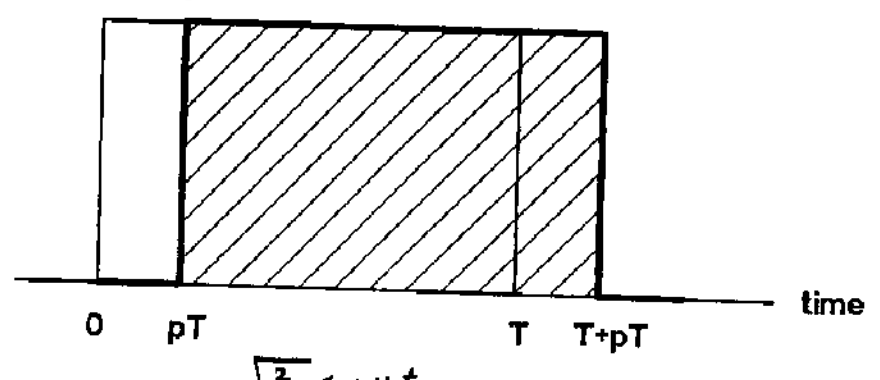
$$= 2Q(10.954 \times 0.383) = 2Q(4.192)$$

$$\cong \frac{2}{4.192\sqrt{2\pi}} \exp\left[-\frac{(4.192)^2}{2}\right] = 2.9 \times 10^{-5}$$

$$P_B \cong \frac{P_E}{k} \cong 9.7 \times 10^{-6}$$

$\therefore$ Choose coherent 8-ary PSK

4.17 (a) The detection of a symbol starts early (late), and concludes early (late) by an amount pT.



$\sqrt{\frac{2}{T}} \cos \omega_1 t$

$s_1(t) = \sqrt{\frac{2E_b}{N_0}} \cos \omega_1 t \longrightarrow \otimes \longrightarrow \int_0^T \longrightarrow z_1(T) = a_1(T) + n_0(T)$

$$a_1(T) = \frac{2\sqrt{E_b}}{T} \int_0^T \cos^2 \omega_1 t \, dt$$

For the received waveform sequence, assume that $s_1(t)$ is followed by $s_2(t) = -\sqrt{\frac{2E_b}{N_0}} \cos \omega_1 t$. Therefore, for the detector late by an amount pT

$$a_1(T) = \frac{2\sqrt{E_b}}{T} \left[\int_{pT}^{T} \cos^2 \omega_1 t \, dt + \int_{T}^{T+pT} -\cos^2 \omega_1 t \, dt \right]$$

$$= \frac{\sqrt{E_b}}{T} \left[T - pT - (T + pT - T) \right] = \sqrt{E_b} \, (1 - 2p)$$

If $s_2(t)$ had been transmitted, followed by $s_1(t)$, then similarly $a_2(T) = -\sqrt{E_b} \, (1-2p)$

Assume that ½ the time, $s_1(t)$ is followed by $s_2(t)$ and ½ the time, $s_1(t)$ is followed by $s_1(t)$. Then,

$$P_B = \frac{1}{2} Q\left(\sqrt{\frac{2E_b}{N_0}}\right) + \frac{1}{2} Q\left[\sqrt{\frac{2E_b}{N_0}} \, (1 - 2p)\right]$$

4.17 (b) When $p = 0$, and $\frac{E_b}{N_0} = 9.6\,dB$ (or 9.12)

$$P_B = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) = Q\,(4.27) = 10^{-5}$$

When $p = 0.2$

$$P_B = \frac{1}{2}Q\left(\sqrt{\frac{2E_b}{N_0}}\right) + \frac{1}{2}Q\left[\sqrt{\frac{2E_b}{N_0}}\,(1-2p)\right]$$

$$= \frac{1}{2} \times 10^{-5} + \frac{1}{2}Q\,(4.27 \times 0.6)$$

$$= 2.6 \times 10^{-3}$$

(c) $P_B = 10^{-5} = \frac{1}{2}Q\left(\sqrt{\frac{2E_b}{N_0}}\right) + \frac{1}{2}Q\left[\sqrt{\frac{2E_b}{N_0}}\,(1-2p)\right]$

By trial-and-error, we can find

$E_b/N_0 = 23.56$ (or 13.7 dB)

Which represents an increase of 4.1 dB needed to restore the $P_B = 10^{-5}$

Check on the trial-and-error result of $E_b/N_0 = 23.56$

$$P_B = \frac{1}{2}Q\left(\sqrt{2 \times 23.56}\right) + \frac{1}{2}Q\left(\sqrt{2 \times 23.56} \times 0.6\right)$$

$$= \frac{1}{2}Q\,(6.865) + \frac{1}{2}Q\,(6.865 \times 0.6)$$

$$= 1.7 \times 10^{-12} + \frac{1}{2}\left(2 \times 10^{-5}\right) \checkmark$$

4.18

$A_1(t) = \sqrt{\frac{2E_b}{T}} \cos \omega_1 t$

For equally-likely signaling, ½ the time

$$a_1(T) = \frac{2}{T}\sqrt{E_b}\left[\int_{pT}^{T} \cos^2 \omega_1 t \, dt - \int_{pT}^{T} \underbrace{\cos \omega_1 t \cos \omega_2 t \, dt}_{0}\right]$$

$$+ \frac{2}{T}\sqrt{E_b}\left[\int_{T}^{T+pT} \underbrace{\cos \omega_2 t \cos \omega_1 t}_{0} \, dt - \int_{T}^{T+pT} \cos^2 \omega_2 t \, dt\right]$$

$$= \sqrt{E_b}\,(1-2p) \qquad \text{Similarly for } a_2(T).$$

When $p = 0$:

$$P_B = Q\left(\sqrt{\frac{E_b}{N_0}}\right) = Q\left(\sqrt{9.12}\right) = Q(3.02) = 1.3 \times 10^{-3}$$

When $p = 0.2$:

$$P_B = \frac{1}{2} Q\left(\sqrt{\frac{E_b}{N_0}}\right) + \frac{1}{2} Q\left(\sqrt{\frac{E_b}{N_0}}\,(1-2p)\right)$$

$$= \frac{1}{2}\left[Q(3.02) + Q(3.02 \times 0.6)\right] = \frac{1}{2}\left[1.3 \times 10^{-3} + 3.5 \times 10^{-2}\right]$$

$= 1.8 \times 10^{-2}$ Or, in terms of additional E_b/N_0 needed to maintain $P_B = 1.3 \times 10^{-3}$

$$P_B = 1.3 \times 10^{-3} = \frac{1}{2}\left[Q\left(\sqrt{\frac{E_b}{N_0}}\right) + Q\left(\sqrt{\frac{E_b}{N_0}} \times 0.6\right)\right]$$

Solving by trial-and-error,

$\frac{E_b}{N_0} = 21.8$ (or 13.4 dB) which represents an increase of 3.8 dB.

4.19 (a) $P_B = \frac{1}{2} Q\left(\sqrt{\frac{2E_b}{N_0}} \cos\phi\right) + \frac{1}{2} Q\left[\sqrt{\frac{2E_b}{N_0}} \cos\phi\,(1-2p)\right]$

(b) $P_B = \frac{1}{2} Q(4.27 \times 0.906) + \frac{1}{2} Q(4.27 \times 0.906 \times 0.6)$

$= \frac{1}{2} Q(3.869) + \frac{1}{2} Q(2.321)$

$= \frac{1}{2}(5.79 \times 10^{-5}) + \frac{1}{2}(0.0102) = \boxed{5.1 \times 10^{-3}}$

(c)

Or, in terms of additional E_b/N_0 in dB that must be provided in order to maintain $P_B = 10^{-5}$

$P_B = 10^{-5} = \frac{1}{2} Q\left(\sqrt{\frac{2E_b}{N_0}}\ 0.906\right) + \frac{1}{2} Q\left(\sqrt{\frac{2E_b}{N_0}}\ 0.544\right)$

$2 \times 10^{-5} \cong Q\left(\sqrt{\frac{2E_b}{N_0}}\ 0.544\right) \cong \frac{1}{x\sqrt{2\pi}} e^{-x^2/2}$

By Trial-and-error, $x = \left(\sqrt{\frac{2E_b}{N_0}}\ 0.544\right) = 4.119$

Thus, $E_b/N_0 = 28.66 = \boxed{14.6 \text{ dB}}$

(an increase of 5 dB over 9.6 dB)

4.20

For the discrete matched filter shown below, note that the symbol $\otimes$ represents an exclusive NOR gate when the signals are binary (1,0) logic levels. The symbol $\otimes$ can also represent waveform multiplication, when the signals are bipolar pulses.

4.20 (cont'd.)

$V_{in}(t)$ Barker coded signal

1 1 1 0 1 — A, −A, t — T, 5T — t_{-4}

1 1 1 0 1 — A, −A, t — t_{-3}

1 1 1 0 1 — A, −A, t — t_{-2}

1 1 1 0 1 — A, −A, t — t_{-1}

1 1 1 0 1 — A, −A, t — t_0

$V_{in}(t)$

Received data → S_1 S_2 S_3 S_4 S_5

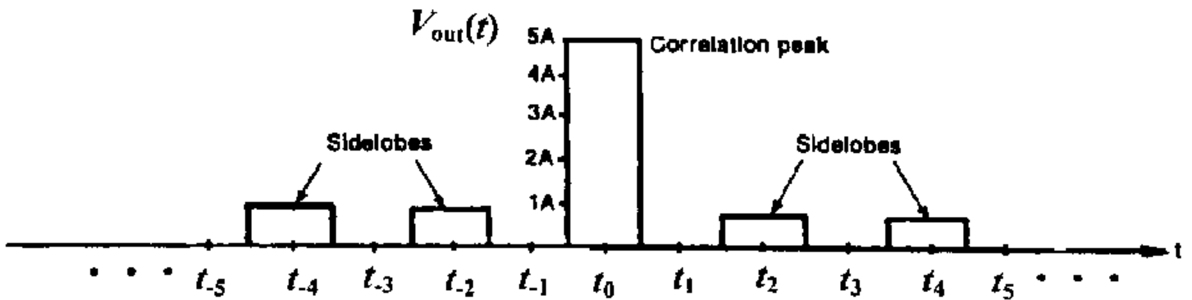
Exclusive NOR gates

Σ Summing circuits → To decision circuit $V_{out}(t)$

R_1 R_2 R_3 R_4 R_5 ← Reference data

1 1 1 0 1

5-b Barker code stored in reference shift registers



Chapter 5

5.1 (a) $\lambda = \frac{c}{f} = \frac{3 \times 10^8 \text{ m/s}}{10^8 \text{ Hz}} = 3 \text{ meters}$

$L_s = \left(\frac{4\pi d}{\lambda}\right)^2$ where $d = 3 \text{ miles} \times 1609 \text{ m/mile}$

$L_s = 4.09 \times 10^8 = 86.1 \text{ dB}$

(b) $P_r = P_t \Big/ \left(\frac{4\pi d}{\lambda}\right)^2$

$= P_t\,(\text{dBW}) - L_s\,(\text{dB}) = -76.1 \text{ dBW}$

(c) $P_r = 13 \text{ dBW} - 86.1 \text{ dB} = -73.1 \text{ dBW}$

(d) $G = \frac{4\pi A_e}{\lambda^2} = k d^2$ where d is the antenna diameter. If the diameter is doubled,

$G' = k(2d)^2 = 4kd^2,$

the gain is increased by a factor of 4, (or 6 dB).

(e) $G = \frac{4\pi A_e}{\lambda^2} = \frac{4\pi \eta A_p}{\lambda^2} = \frac{\pi^2 \eta d^2}{\lambda^2}$

$d^2 = \frac{\lambda^2 G}{\pi^2 \eta} = \frac{(3 \text{ meter})^2 \times 10}{\pi^2 \times 0.55} = 16.58 \text{ m}^2$

$d = \sqrt{16.58} = 4.07 \text{ meters}$

$= 13.35 \text{ feet}$

5.2 (a) $G = \dfrac{4\pi A_e}{\lambda^2} = \dfrac{4\pi \eta A_p}{\lambda^2}$

$$= \frac{\pi^2 d^2 \eta \times (2\times10^9)^2}{(3\times10^8)^2} \quad \text{where } d = 3 \text{ ft.}$$

$$d = 3 \text{ ft} \times 0.3048 \text{ m/ft} = 0.9144 \text{ m}$$

$$G = 201.7 = 23 \text{ dB}$$

(b) $EIRP = P_t G_t = 2 \times 201.7 = 403.4$

$$= 26 \text{ dBW}$$

(c) $L_s = \left(\dfrac{4\pi d}{\lambda}\right)^2 = \left(\dfrac{4\pi \times 25 \text{ miles} \times 1609 \text{ m/mile}}{3\times10^8 / 2\times10^9}\right)^2$

$$= 1.13 \times 10^{13} = 130.5 \text{ dB}$$

$$P_r = \frac{EIRP\ G_r}{L_s} = EIRP\ (\text{dBW}) + G_r(\text{dB}) - L_s(\text{dB})$$

$$= 26 + 23 - 130.5 = -81.5 \text{ dBW}$$

5.3 $M = \dfrac{EIRP\ G/T_\circ}{(E_b/N_0)\, R\, k\, L_s\, L_o}$; $M = 1$

$$G_r = \left[(E_b/N_0)\cdot R \cdot k \cdot L_s \cdot L_o \cdot T_s^\circ\right] / EIRP$$

$$G_r\ (\text{dB}) = 10 + 76.99 - 228.6 + 206.1 + 27.78 - 57$$

$$= 35.27 \text{ dB} = 3365.12$$

$$A_p = \frac{\lambda^2 G_r}{4\pi\eta} = \frac{(3\times10^8/12.5\times10^9)^2 \times 3365}{4\pi \times 0.55}$$

$= 0.28 \text{ m}^2$; diameter $= 0.598$ meter (approx. 2 ft.)

5.4 $P_{out} = \frac{V_o^2}{R} = k\left(T_o^\circ + T_R^\circ\right) W G$

$$= \frac{10^{-8}}{50} = 1.38\times10^{-23}\left(290 + T_R^\circ\right) 10^4 \times 10^6$$

$$T_R^\circ + 290 = \frac{2\times 10^{-20}}{1.38\times10^{-23}}$$

$$T_R^\circ = 1159\ \mathrm{K}$$

5.5 $N_i = kT^\circ W = 1.38\times10^{-23} \times 290 \times 5\times10^5$

$$= 2\times10^{-15}\ W$$

$F = 4\ dB = 2.51 = (S/N)_i / (S/N)_o$

Let $(S/N)_o = 1$; $(S/N)_i = 2.51$

$$S_i = 2.51\ N_i = 5.02\times10^{-15}\ W$$

$$\frac{x_{in}^2}{R} = 5.02\times10^{-15}\ W$$

$$x_{in}^2 = 2.51\times10^{-13}\ \text{volt}^2$$

$$x_{in} = 0.5\ \mu V$$

5.6 (a) $P_B = 10^{-3} = Q\left(\sqrt{\frac{2E_b}{N_o}}\right)$

From Table B.1, $\sqrt{\frac{2E_b}{N_o}} = 3.09$

$$\frac{E_b}{N_o} = 4.77 = 6.8\ dB$$

$$L_s = \left(\frac{4\pi d}{\lambda}\right)^2 = \left(\frac{4\pi \times 4\times 10^7}{3\times 10^8 / 3\times 10^9}\right)^2$$

$$= 194 \text{ dB}$$

$$N_0 = kT^\circ = \frac{EIRP \; G_r}{M\,(E_b/N_0)_{reqd}\; R\, L_s}$$

$$kT^\circ (\text{dBW/Hz}) = 20 + 10 - (3 + 6.8 + 20 + 194)$$

$$kT^\circ = -193.8 \text{ dBW/Hz}$$

$$= 4.17 \times 10^{-20} \text{ watt/Hz}$$

(b) $T_s^\circ = N_0/k = \dfrac{4.17\times 10^{-20}}{1.38\times 10^{-23}} = 3022 \text{ K}$

$T_R^\circ = T_s^\circ - T_A^\circ = 3022 - 290 = 2732 \text{ K}$

(c) $T_R^\circ = (F-1)\,290$

$F = \dfrac{T_R^\circ}{290} + 1 = \dfrac{2732}{290} + 1 = 10.42 = 10.2 \text{ dB}$

5.7 (a) $T_R^\circ = (F-1)\,290 = (20-1)\,290$

$= 5510 \text{ K}$

(b) $T_s^\circ = T_A^\circ + T_R^\circ = 490 + 5510 = 6000 \text{ K}$

(c) $N_{out} = GkT_s W = 10^6 \times 1.38\times 10^{-23} \times 6000 \times 2\times 10^6$

$= 1.66 \times 10^{-7} \text{ watt}$

$$S_{out} = G S_{in} = 10^6 \times 10^{-12}$$
$$= 10^{-6} \text{ watt}$$
$$\left(S/N\right)_{out} = \frac{10^{-6}}{1.66 \times 10^{-7}} = 6.02 = 7.8\,dB$$

5.8

$T_A^o = 10\,K$ → [G = 50 dB, F = 10 dB, W = 500 MHz] → $S_{out} = 50 \times 10^{-12} \times 10^5 = 5\,\mu W$

$S_{in} = 50 \times 10^{-12}\,W$

WITHOUT PREAMP: $T_R^o = (F-1)\,290 = 2610\,K$

$T_S^o = T_A^o + T_R^o = 10 + 2610 = 2620\,K$

$N_{out} = G k T_S^o W = 10^5 \times 1.38 \times 10^{-23} \times 2620 \times 500 \times 10^6$
$= 1.81\,\mu W$

$(S/N)_{out} = 5\,\mu W / 1.81\,\mu W = 2.76$

Requried improvement is 10 dB, ∴ need
$(S/N)_{out} = 27.6$

WITH PREAMP:

$T_A^o = 10\,K$ → [G = 20 dB, F = ?, W = 500 MHz] → [G = 50 dB, F = 10 dB, W = 500 MHz] →

$S_{out} = 50 \times 10^{-12} \times 10^2 \times 10^5 = 500\,\mu W$

$$\left(S/N\right)_{out} = 27.6 = \frac{500\,\mu W}{N_{out}}$$

$$N_{out} = \frac{500\,\mu W}{27.6} = 18.12\,\mu W$$ {permissable noise power out}

$$N_{out} = G k T_s^\circ W = 18.12\ \mu W$$

$$T_s^\circ = \frac{18.12 \times 10^{-6}}{10^{2} \times 10^{5} \times 1.38 \times 10^{-23} \times 500 \times 10^{6}}$$

$$= 262.6\ K$$

$$T_{COMP}^\circ = T_{R1}^\circ + \frac{T_{R2}^\circ}{G_1} = 252.6\ K$$

$$T_{R1}^\circ = 252.6 - \frac{2610}{100} = 226.5\ K$$

$$T_{R1}^\circ = (F_1 - 1)\,290; \quad F_1 = \frac{226.5}{290} + 1$$

$$F_1 = 1.78 = 2.5\ dB$$

5.9 $$P_B = \frac{1}{2} e^{-\frac{1}{2}\frac{E_b}{N_0}} = 10^{-5}$$

$$E_b/N_0 = -2 \ln\left(2 \times 10^{-5}\right) = 21.64$$

$$= 13.4\ dB$$

$$L_s = \left(\frac{4\pi d}{\lambda}\right)^2 = \left(\frac{4\pi \times 10^5}{3 \times 10^8 / 12 \times 10^9}\right)^2 = 154\ dB$$

$$T_s^\circ(dBK) = EIRP + G_r - \left(M + \frac{E_b}{N_0} + R + k + L_s + L_0\right)$$

$$= 10 + 0 - (0 + 13.4 + 40 - 228.6 + 154)$$

$$= 31.2\ dBK$$

$$= 1318\ K$$

5.10

→ [$G_1 = 20\,dB$, $F_1 = 6\,dB$] → [3 dB Loss] → [$G_3 = 60\,dB$, $F_3 = 16\,dB$] →

(a) $$F = F_1 + \frac{F_2 - 1}{G_1} + \frac{F_3 - 1}{G_1 G_2}$$

$$F = 4 + \frac{2-1}{100} + \frac{39.8 - 1}{100 \times 1/2}$$

$$= 4.79 = 6.8\,dB$$

(b) $$F = 2 + \frac{39.8 - 1}{1/2}$$

$$= 79.6 = 19\,dB$$

5.11

(a) $$T_R^\circ = T_{R1}^\circ + \frac{T_{R2}^\circ}{G_1} + \frac{T_{R3}^\circ}{G_1 G_2}$$

$$= 1800 + \frac{2700}{10} + \frac{4800}{10 \times 39.81} = 2082\ K$$

(b) 10% of 1800 K = 180K

$$180 = \frac{2700}{G_1} + \frac{4800}{39.81\,G_1}$$

$$G_1 = \frac{2700}{180} + \frac{4800}{180 \times 39.81} = 15.67$$

$$= 12\ dB$$

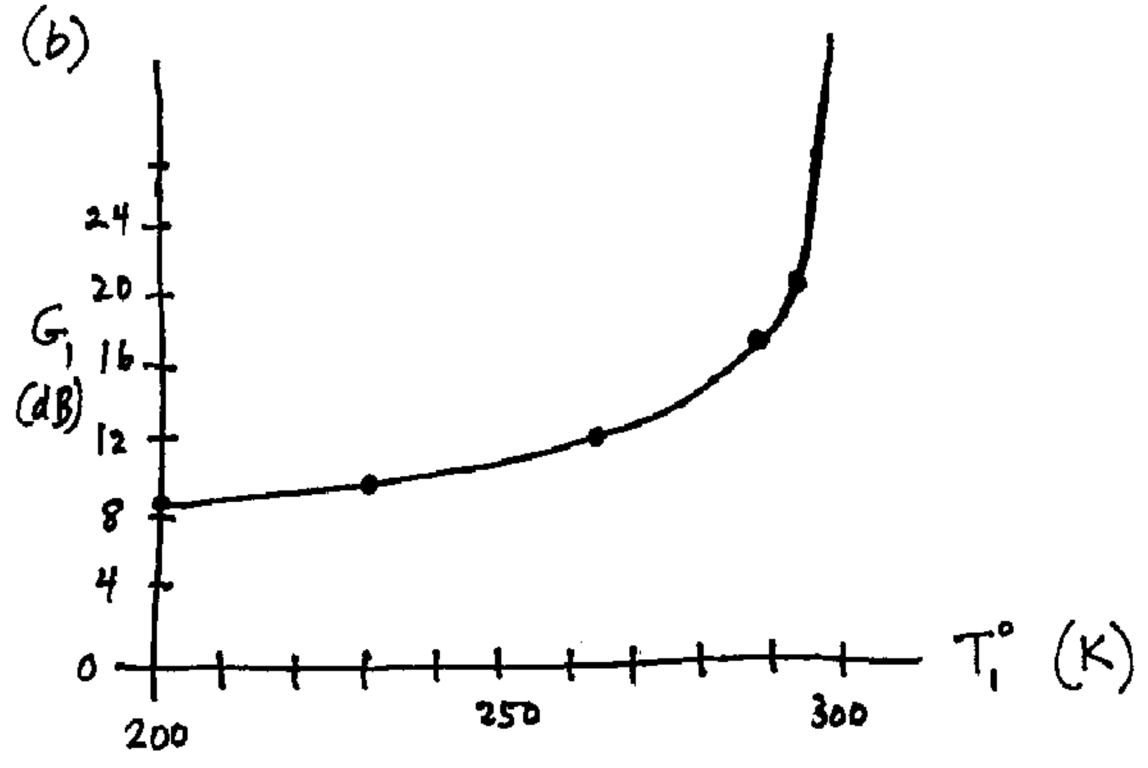
5.12 (a) $T_R^\circ = T_1^\circ + \frac{T_2^\circ}{G_1} + \frac{T_3^\circ}{G_1 G_2} + \frac{T_4^\circ}{G_1 G_2 G_3}$

$$300 = T_1^\circ + \frac{1}{G_1}\left(600 + \frac{2000}{20} + \frac{2000}{20\times 100}\right)$$

$$300 \cong T_1^\circ + \frac{700}{G_1} \; ; \quad G_1 = \frac{700}{300 - T_1^\circ}$$

T_1°	G_1	G_1 in dB
200	7	~ 8.5
230	10	10
265	20	13
290	70	~ 18.5
295	140	~ 21.5
300	∞	

(b)



(c) T_5° contribution to T_R° is $\frac{T_5^\circ}{G_1 G_2 G_3 G_4}$

From part (a), G_1 is at least equal to 7, the T_5° contribution is $\frac{T_5^\circ}{7 \times 20 \times (100)^2} = \frac{T_5^\circ}{1.4 \times 10^6}$

Thus, even if T_5° were equal to 10^6 K, it would contribute less than 1° to the composite T_R°.

(d) For this example, G_1 drops slowly for values of T_1° below 230 K, and G_1 increases rapidly for values of T_1° above 290 K, therefore consider choosing T_1° in the range of 230 to 290 K.

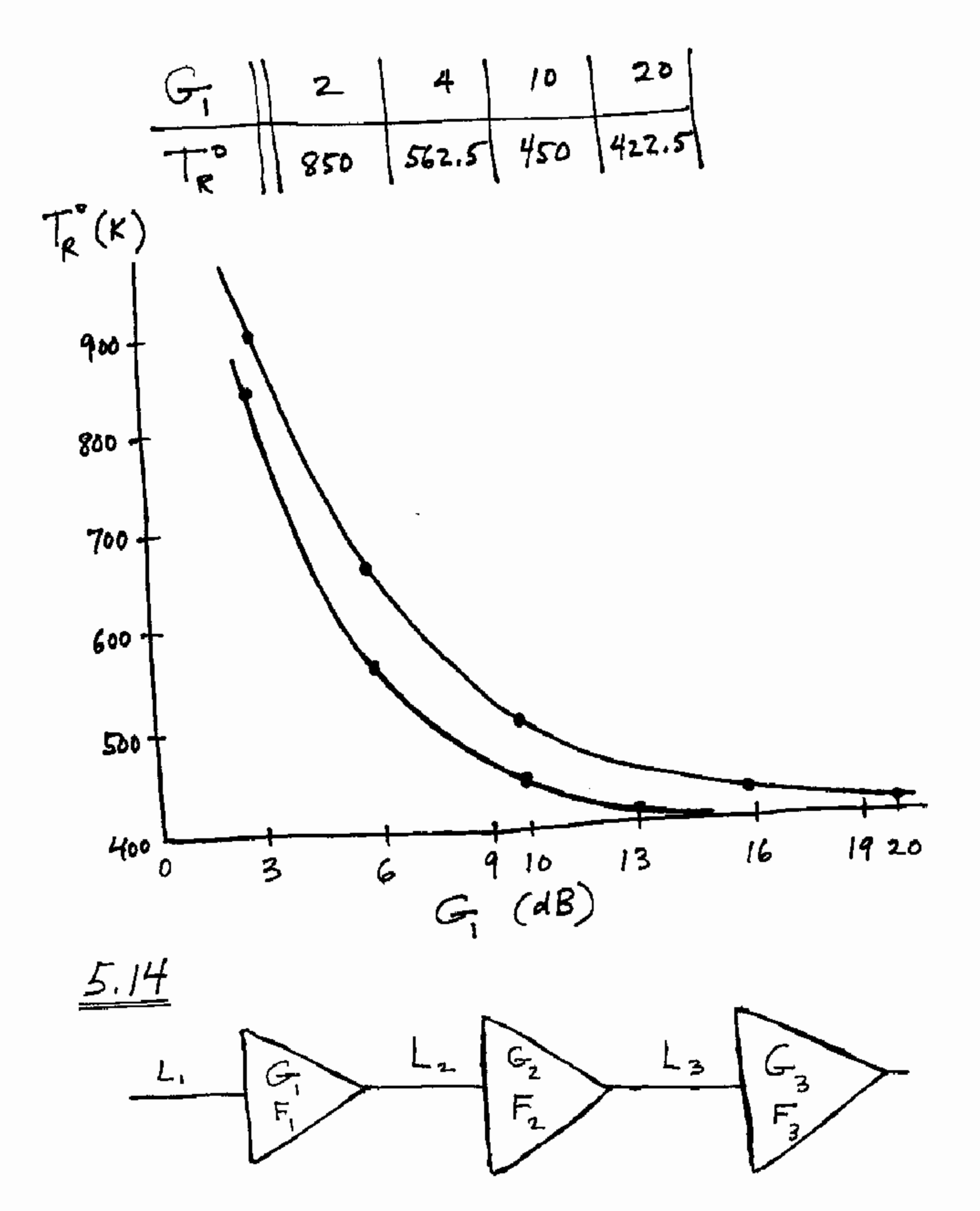
5.13 (a) $T_R^\circ = T_1^\circ + \frac{T_2^\circ}{G_1} = 400 + \frac{1000}{G_1}$

G_1	2	4	10	40	100
T_R°	900	650	500	425	410

(b) $T_R^\circ = T_1^\circ + \frac{T_1^\circ}{G_1} + \frac{T_2^\circ}{G_1^2}$

$$= 400 + \frac{400}{G_1} + \frac{1000}{G_1^2}$$

G_1	2	4	10	20
T_R°	850	562.5	450	422.5

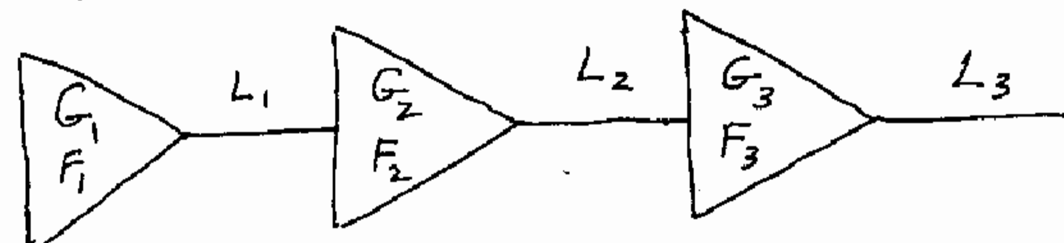


5.14

$$F_{comp} = L_1 + L_1(F_1-1) + \frac{L_1(L_2-1)}{G_1}$$

$$+ \frac{L_1 L_2 (F_2-1)}{G_1} + \frac{L_1 L_2 (L_3-1)}{G_1 G_2} + \frac{L_1 L_2 L_3 (F_3-1)}{G_1 G_2}$$

5.14 (b)



$$F_{comp} = F_1 + \frac{L_1 - 1}{G_1} + \frac{L_1(F_2-1)}{G_1} + \frac{L_1(L_2-1)}{G_1 G_2}$$

$$+ \frac{L_1 L_2 (F_3-1)}{G_1 G_2} + \frac{L_1 L_2 (L_3-1)}{G_1 G_2 G_3}$$

5.14 (c)

$N_i \rightarrow$ [NETWORK G] $\rightarrow N_{out}$

In general, the noise power output of a network is computed as follows (where G is the network gain).

$$N_{out} = \underbrace{G \times N_i}_{\text{contribution due to the input noise source}} + \underbrace{G(F-1)\,kT_0W}_{\text{contribution due to network noise}}$$

contribution due to network noise expressed in terms of noise figure F and a 290K noise reference.

5.14 (c) (cont'd.)

$T_A = 1160K$

$L_1 = 6dB$ — $G_1 = 13dB$, $F_1 = 3dB$ — $L_2 = 10dB$ — $G_2 = 10dB$, $F_2 = 6dB$

Test points: 01, 02, 03, 04

$S_i = 8\times10^{-11} W$

Bandwidth $W = 0.25 GHz$

$kT_oW = 10^{-12}$

where $T_o = 290K$

$$(SNR)_{input} = \frac{S_i}{N_i} = \frac{S_i}{kT_A W} = \frac{8\times10^{-11} W}{4\times10^{-12} W} = 20$$

$$N_{01} = \frac{kT_A W}{L_1} + \frac{L_1 - 1}{L_1} kT_o W$$

where the gain of the lossy line is $1/L_1$

$$N_{01} = \frac{4\times10^{-12}}{4} + \frac{3}{4}10^{-12} = 1.75\times10^{-12}$$

$$S_{01} = \frac{8\times10^{-11}}{4} = 2\times10^{-11}$$

$$SNR_{01} = \frac{S_{01}}{N_{01}} = \frac{2\times10^{-11}}{1.75\times10^{-12}} = 11.4$$

$$N_{02} = G_1 N_{01} + G_1 (F_1 - 1) kT_o W$$
$$= 20\times1.75\times10^{-12} + 20(2-1)10^{-12}$$
$$= 5.5\times10^{-11}$$

$$S_{02} = G_1 S_{01} = 20\times2\times10^{-11} = 4\times10^{-10}$$

$$SNR_{02} = \frac{S_{02}}{N_{02}} = \frac{4\times10^{-10}}{5.5\times10^{-11}} = 7.3$$

5.14 (c) (cont'd.)

$$N_{o3} = \frac{N_{o2}}{L_2} + \frac{(L_2 - 1) kT_o W}{L_2}$$

$$= \frac{5{,}5 \times 10^{-11}}{10} + \frac{(10-1)\, 10^{-12}}{10}$$

$$= 5.5 \times 10^{-12} + 0.9 \times 10^{-12}$$

$$= 6{,}4 \times 10^{-12}$$

$$S_{o3} = \frac{S_{o2}}{L_2} = \frac{4 \times 10^{-10}}{10} = 4 \times 10^{-11}$$

$$SNR_{o3} = \frac{S_{o2}}{N_{o2}} = \frac{4 \times 10^{-11}}{6{,}4 \times 10^{-12}} = 6.25$$

$$N_{o4} = G_2 N_{o3} + G_2 (F_2 - 1) kT_o W$$

$$= 10 \times 6{,}4 \times 10^{-12} + 10\,(4-1)\, 10^{-12}$$

$$= 6.4 \times 10^{-11} + 3 \times 10^{-11}$$

$$= 9.4 \times 10^{-11}$$

$$S_{o4} = G_2\, S_{o3} = 10 \times 4 \times 10^{-11} = 4 \times 10^{-10}$$

$$SNR_{o4} = \frac{S_{o4}}{N_{o4}} = \frac{4 \times 10^{-10}}{9.4 \times 10^{-11}}$$

$$= 4.26$$

5.15 (a)

$S_i = 10^{-11}$ W → [G = 10 dB, F = 3 dB] → (01) — L = 10 dB → (02)

$T_A = 290$ K

$W = 0.25$ GHz

$N_i = kT_A W = 10^{-12}$ W $\quad (SNR)_{input} = \frac{S_i}{N_i}$

$$= \frac{10^{-11}}{10^{-12}} = 10$$

$$N_{o1} = GN_i + G(F-1)kT_o W$$
$$= 10 \times 10^{-12} + 10(2-1)10^{-12}$$
$$= 2 \times 10^{-11}$$

$$S_{o1} = GS_i = 10 \times 10^{-11} = 10^{-10}$$

$$SNR_{o1} = S_{o1}/N_{o1} = 10^{-10}/2\times10^{-11} = 5$$

$$N_{o2} = \frac{N_{o1}}{L} + \frac{(L-1)\,kT_o W}{L}$$
$$= \frac{2\times10^{-11}}{10} + \frac{9\times10^{-12}}{10} = 2.9\times10^{-12}$$

$$S_{o2} = S_{o1}/L = 10^{-10}/10 = 10^{-11}$$

$$SNR_{o2} = S_{o2}/N_{o2} = 10^{-11}/2.9\times10^{-12} = 3.45$$

(b) Repeat for $T_A = 1450\text{ K} = 5 \times T_o$

$$(SNR)_{input} = \frac{S_i}{N_i} = \frac{10^{-11}}{5\times10^{-12}} = 2$$

5.15 (b) (cont'd.)

$$N_{o1} = GN_i + G(F-1)\,kT_o W$$
$$= 10 \times 5\times10^{-12} + 10\,(2-1)\,10^{-12}$$
$$= 5\times10^{-11} + 10^{-11} = 6\times10^{-11}$$
$$S_{o1} = GS_i = 10 \times 10^{-11} = 10^{-10}$$
$$SNR_{o1} = S_{o1}/N_{o1} = 10^{-10}/6\times10^{-11} = 1.67$$

$$N_{o2} = \frac{N_{o1}}{L} + \frac{(L-1)\,kT_oW}{L}$$
$$= \frac{6\times10^{-11}}{10} + 0.9\times10^{-12} = 6.9\times10^{-12}$$
$$S_{o2} = \frac{S_{o1}}{L} = \frac{10^{-10}}{10} = 10^{-11}$$
$$SNR_{o2} = \frac{S_{o2}}{N_{o2}} = \frac{10^{-11}}{6.9\times10^{-12}} = 1.45$$

5.16 (a) $N_i = kT_A^{\circ}W$
$$= 1.38\times10^{-23}\times 600 \times 40\times10^{6}$$
$$= 3.3\times10^{-13}\ W$$

(b) $N_{ai} = kT_R^{\circ}W$
$$= 1.38\times10^{-23}\times 3000 \times 40\times10^{6} = 1.66\times10^{-12}\ W$$

(c) $N_{out} = G(N_i + N_{ai})$
$$= 10^{8}(3.3\times10^{-13} + 1.66\times10^{-12}) = 199\ \mu W$$

5.17 (a) $T_R^\circ = (F-1)\,290^\circ$; $F = 2\,dB = 1.58$

$T_R^\circ = (1.58-1)\,290^\circ = 168.2\ K$

(b) $N_{out} = k(T_A^\circ + T_R^\circ)\,G\,W$

$= 1.38 \times 10^{-23}(50 + 168.2) \times 1000 \times 20 \times 10^6$

$= 6.02 \times 10^{-11}\ W$

$S_{out} = 10^{-12} \times 1000 = 10^{-9}\ W$

$SNR = 10^{-9} / 6.02 \times 10^{-11} = 16.6 = 12.2\ dB$

5.18 (a)

$F = 3\,dB/100\,ft \times 75\,ft = 2.25\ dB$

$L = 2.25\ dB = 1.68$

$F_{COMP} = F + L(F-1) = 1.68 + 1.68(19.95-1)$

$= 33.52 = 15.25\ dB$

5.18 (b)

(b)

LOSSY LINE $F_1 = 2.25\,dB$ → PREAMP $G_2 = 20\,dB$, $F_2 = 3\,dB$ → RECEIVER $F_3 = 13\,dB$ →

$$F_{COMP} = F_1 + \frac{F_2 - 1}{G_1} + \frac{F_3 - 1}{G_1 G_2}$$

$$= 1.68 + 1.68(2-1) + \frac{1.68(19.95-1)}{100}$$

$$= 3.68 = 5.66\ dB$$

(c) $$F_{COMP} = 2 + \frac{1.68-1}{100} + \frac{1.68(19.95-1)}{100}$$

$$= 2.325 = 3.66\ dB$$

5.19 (a) DPSK: $P_B = 10^{-5} = \frac{1}{2} e^{-E_b/N_0}$

$$\ln 2 \times 10^{-5} = -E_b/N_0$$

$$E_b/N_0 = 10.82 = 10.34\ dB$$

$$G_t = \frac{4\pi A_e}{\lambda^2} = \frac{\pi^2 d^2 \eta f^2}{c^2}$$

$d = 2\ ft \times 0.3048\ m/ft = 0.61\ m$

$$G_t = \frac{\pi^2 (0.61)^2 \times 0.55 \times (8 \times 10^9)^2}{(3 \times 10^8)^2} = 1436$$

$EIRP = 20\ W \times 1436 = 28720 = 44.6\ dBW$

RECEIVING ANTENNA: $8\ ft. \times 0.3048\ m/ft = 2.438\ m$

$$G_r = \frac{\pi^2 (2.438)^2 \times 0.55 \times (8 \times 10^9)^2}{(3 \times 10^8)^2} = 43.6\ dB$$

$T_s^\circ = 100\ K$; $G/T^\circ = 43.6 - 20 = 23.6\ dB/K$

$L_s = \left(\frac{4\pi d}{\lambda}\right)^2$; $d = 20{,}000$ n miles $\times 1852$ m/n mile $= 3.7 \times 10^7\ m$

$$L_s = \left(\frac{4\pi \times 3.7 \times 10^7 \times 8 \times 10^9}{3 \times 10^8}\right)^2 = 1.537 \times 10^{20}$$

$= 201.9\ dB$

$$R = \frac{EIRP\ G/T^\circ}{M\,(E_b/N_0)_{reqd}\ k\, L_s L_o}$$

where $k = 1.38 \times 10^{-23}$ $= -228.6\ dB/K$

$R(dB\text{-}Hz) = 44.6 + 23.6 - 10.3 + 228.6 - 201.9 - 2$
$= 82.6\ dB\text{-}Hz = 182$ Mbits/s.

(b) Downlink frequency is 2×10^9 Hz:
Parameters affected are: EIRP, G/T, and L_s.
EIRP is reduced by a factor of $(4)^2 = 16$, a reduction of 12.04 dB.

Similarly, G/T_0 and L_s are each reduced by 12.04 dB. Hence, at 2 GHz, the maximum data rate is

$$R(\text{dB-Hz}) = 32.56 + 11.56 - 10.3 + 228.6 - 189.86 - 2$$
$$= 70.6 \text{ dB-Hz} = 11.5 \text{ Mbits/s.}$$

5.20 $L_s = \left(\frac{4\pi d}{\lambda}\right)^2$; $d = 7.9 \times 10^8$ miles $\times 1609$ m/mile $= 1.27 \times 10^{12}$ m

$$L_s = \left(\frac{4\pi \times 1.27 \times 10^{12} \times 2 \times 10^9}{3 \times 10^8}\right)^2 = 280.5 \text{ dB}$$

$$G_r = \frac{\pi^2 d^2 \eta \times (2 \times 10^9)^2}{(3 \times 10^8)^2}; \quad d = 75 \text{ ft} \times 0.3048 \text{ m/ft} = 22.86 \text{ m}$$

$$G_r = \frac{\pi^2 (22.86)^2 \times 0.55 \times 4 \times 10^{18}}{9 \times 10^{16}} = 51 \text{ dB}$$

$T_s^\circ = 20\text{K} = 13 \text{ dBK}$; $G/T = 51 - 13 = 38$ dB/K

$$EIRP = \frac{(E_b/N_0)_{reqd} R k L_s L_0}{G/T}$$

$$EIRP(\text{dBW}) = 10 - 228.6 + 20 + 280.5 + 3 - 38$$
$$= 46.9 \text{ dBW} = 48978.\ \text{W}$$

$$G_t = \frac{EIRP}{P_t} = \frac{48978.}{10} = 4898.$$

$$G_t = \frac{4\pi A_e}{\lambda^2} = \frac{\pi^2 d^2 \eta f^2}{c^2}$$

$$d^2 = \frac{G_t\, c^2}{\pi^2 \eta f^2} = \frac{4898 \times (3\times 10^8)^2}{\pi^2 \times 0.55 \times (2\times 10^9)^2}$$

$$= 20.3\;;\quad d = 4.5\, m \times \left(\frac{1}{0.3048}\, ft/m\right)$$

Transmitting antenna diam. = 14.8 ft

5.21

(a) $F_{COMP} = F_1 + \frac{F_2 - 1}{G_1} = 1.259 + \frac{4-1}{10} = 1.559$

$F_{COMP} = 1.93\, dB$; IMPROVEMENT $\cong$ 4.1 dB

(b) <u>SNR at output before preamp</u>:

$N_{out} = Gk(T_A^\circ + T_R^\circ)W = Gk(T_A^\circ + (F-1)290)W$

$= 10^6 \times 1.38 \times 10^{-23} \times [290 + (4-1)\,290] \times 5\times 10^8$

$= 8\,\mu W$

$SNR_{out} = 64\,\mu W / 8\,\mu W = 8 = 9\, dB$

<u>SNR at output after preamp</u>:

$N_{out} = 10^7 \times 1.38\times 10^{-23} \times [290 + (1.559-1)\,290]\, 5\times 10^8$

$= 31.2\,\mu W$

$SNR_{out} = 640\,\mu W / 31.2\,\mu W = 20.5 = 13.1\, dB$

SNR IMPROVEMENT = 13.1 − 9 = 4.1 dB

(c) $T_A^\circ = 6000\,K$

<u>SNR_{out} before preamp:</u>

$N_{out} = 10^6 \times 1.38 \times 10^{-23} \times \left[6000 + (4-1)290\right] \times 5 \times 10^8$

$= 47.4\,\mu W$

$SNR_{out} = 64\,\mu W / 47.4\,\mu W = 1.35 = 1.3\,dB$

<u>SNR_{out} after preamp:</u>

$N_{out} = 10^7 \times 1.38 \times 10^{-23} \times \left[6000 + (1.559-1)290\right] \times 5 \times 10^8$

$= 425.2\,\mu W$

$SNR_{out} = 640\,\mu W / 425.2\,\mu W = 1.51 = 1.8\,dB$

SNR IMPROVEMENT = 1.8 - 1.3 = 0.5 dB

(d) $T_A^\circ = 15\,K$

<u>SNR_{out} before preamp:</u>

$N_{out} = 10^6 \times 1.38 \times 10^{-23} \times \left[15 + (4-1)290\right] \times 5 \times 10^8$

$= 6.1\,\mu W$

$SNR_{out} = 64\,\mu W / 6.1\,\mu W = 10.5 = 10.2\,dB$

<u>SNR_{out} after preamp:</u>

$N_{out} = 10^7 \times 1.38 \times 10^{-23} \times \left[15 + (1.559-1)290\right] \times 5 \times 10^8$

$= 12.2\,\mu W$

$SNR_{out} = 640\,\mu W / 12.2\,\mu W = 52.46 = 17.2\,dB$

SNR IMPROVEMENT = 17.2 - 10.2 = 7 dB

(e) When the predominant contribution of system noise is introduced from the environment outside the system (i.e., $T_A^\circ = 6000\,K$), using a preamplifier with improved noise figure will provide very little improvement to the output SNR. However, when the predominant contribution of the system noise is introduced by the system receiver, using a preamplifier with improved noise figure can provide a great deal of improvement.

5.22 (a) $G_r = \dfrac{4\pi A_e}{\lambda^2} = \dfrac{\pi^2 d^2 (0.55) \times (12\times10^9)^2}{(3\times10^8)^2}$

diam, $d = 0.1$; $G_r = 86.89 = 19.39\ dB$

$L_s = \left(\dfrac{4\pi d}{\lambda}\right)^2 = \left(\dfrac{4\pi \times 10^4 \times 12\times10^9}{3\times10^8}\right)^2 = 134\ dB$

$$M = \frac{EIRP\ G/T^\circ}{(E_b/N_0)_{reqd}\ R\ k\ L_s L_0}$$

$$\begin{aligned} T_s^\circ(dBK) &= EIRP + G_r - E_b/N_0 - R - k - L_s - L_0 \\ &= 0 + 19.39 - 9.6 - 70 + 228.6 - 134 \\ &= 34.39\ dBK = 2747\ K \end{aligned}$$

(where $E_b/N_0 = 9.6\ dB$ for $P_B = 10^{-5}$ is a well-known benchmark for matched filter detection of BPSK).

$$T_R^\circ = T_S^\circ - T_A^\circ = 2747 - 800 = 1947\text{K}$$
$$= (F-1)\,290\text{K} = 1947\text{K}$$
$$F = 7.71 = 8.87 \text{ dB}$$

(b) If the data rate is doubled:

$$T_S^\circ = 31.39 \text{ dBK} = 1377\text{ K}$$
$$T_R^\circ = 1377 - 800 = 577\text{ K}$$
$$F = \frac{577}{290} + 1 = 2.99 = 4.76 \text{ dB}$$

(c) If the antenna diameter is doubled:

$$G_r = (19.39 + 6)\text{ dBi} \quad \left(\text{area is 4 times larger}\right)$$
$$T_S^\circ = (34.39 + 6)\text{ dBK} = 10{,}939\text{K}$$
$$F = \frac{10139}{290} + 1 = 35.96 = 15.56 \text{ dB}$$

5.23 (a) $A_1 P_1 = -130 \text{ dBW}$

$$P_T = \sum_{i=1}^{10} A_i P_i = -120 \text{ dBW}$$

(b) $N_s W = kT^\circ W = 1.38\times10^{-23} \times 2000 \times 5\times10^{7}$

$$= 1.38 \times 10^{-12} \text{ W} = -118.6 \text{ dBW}$$

(c) $SNR_{in} = -130 \text{ dBW} + 118.6 \text{ dBW} = -11.4 \text{dB}$

(d) $$\frac{A_1 P_1}{P_T + N_s W} = \frac{10^{-13}}{10^{-12} + 1.38 \times 10^{-12}}$$
$$= 0.042$$

$EIRP_s = 1000$ W. Therefore, the apportionment for each user is

$$EIRP_s \left(\frac{A_1 P_1}{P_T + N_s W} \right) = 42 \text{ Watts}$$

(e) $$EIRP_s \left(\frac{N_s W}{P_T + N_s W} \right) = \frac{1000 \times 1.38 \times 10^{-12}}{10^{-12} + 1.38 \times 10^{-12}}$$
$$= 580 \text{ Watts}$$

or, alternatively,

$$EIRP_s - 10 \times 42 \text{ watts} = 1000 - 420$$
$$= 580 \text{ watts}$$

(f) Uplink limited.

(g) $$\left(\frac{P_r}{N_0} \right)_1 = \frac{EIRP_s \, \gamma_1 \, \beta A_1 P_1}{EIRP_s \, \gamma_1 \, \beta N_s + N_g}$$

$$= \frac{1000 \times 10^{-14} \left(\dfrac{10^{-13}}{10^{-12} + 1.38 \times 10^{-12}} \right)}{\dfrac{1000 \times 10^{-14} \times 1.38 \times 10^{-23} \times 2000}{10^{-12} + 1.38 \times 10^{-12}} + 1.38 \times 10^{-23} \times 800}$$

$$= 3.3 \times 10^{6} = 65.2 \text{ dB}$$

(h) $$\left(\frac{P_r}{N_0}\right)_1 \cong \frac{A_1 P_1}{N_s} = \frac{10^{-13}}{1.38\times10^{-23}\times 2000}$$

$$= 3.6\times10^6 = 65.6 \text{ dB}$$

(i) $$\left(\frac{P_r}{N_0}\right)^{-1}_{OVERALL} = \left(\frac{P_r}{N_0}\right)^{-1}_{U/L} + \left(\frac{P_r}{N_0}\right)^{-1}_{D/L}$$

$$= \frac{1}{3.6\times10^6} + \frac{1.38\times10^{-23}\times 800}{42\times10^{-14}}$$

$$\left(\frac{P_r}{N_0}\right)_{OVERALL} = 3.3\times10^6 = 65.2 \text{ dB}$$

5.24 $$\frac{G_r}{L_s L_0}\left(\frac{A_1 P_1}{P_T + N_s W}\right) = \frac{50}{5000} = 0.01$$

$$\frac{10\times10^{-14}}{x\cdot 10\times10^{-14} + N_s W} = 0.01$$

$$N_s W = 1.38\times10^{-23}\times 3500\times10^8 = 4.83\times10^{-12} W$$

$$x\cdot10^{-13} + 4.83\times10^{-12} = \frac{10^{-13}}{0.01} = 10^{-11}$$

$$10^{-13}x = 10^{-11} - 4.83\times10^{-12}$$

$$x = \frac{5.17\times10^{-12}}{10^{-13}} = 51.7$$

∴ 51 users can simultaneously use the repeater.

5.25

$$M = \frac{EIRP \;\; G/T}{(E_b/N_0)_{reqd}\, R\, k\, L_s\, L_o}$$

SET M AND L_o EQUAL TO 1. SOLVE FOR SYSTEM TEMP

$$T_s = EIRP + G_r - [(E_b/N_0)_{reqd} + R + k + L_s]$$

EACH IN dB OR dBW

$$G_r = G_t = \frac{4\pi A_e}{\lambda^2} = \frac{4\pi \times \eta \times \pi r^2}{(c/f_0)^2} = 21.713 = 13.4\ dB$$

$$EIRP = -30\ dBW + 13.4\ dB = -16.6\ dBW$$

$$L_s = \left(\frac{4\pi d}{\lambda}\right)^2 = \left(\frac{4\pi \times 100 \times 10^3}{3 \times 10^8/3 \times 10^8}\right)^2$$

$$= 1.579 \times 10^{12} = 122\ dB$$

$(E_b/N_0)_{reqd} = 9.6 dB$ LOSS FACTOR, $L = 1\ dB$

(BOTH GIVEN)

$$T_s = EIRP + G_r - [(E_b/N_0)_{reqd} + R + k + L_s]$$

$$T_s = -16.6 + 13.4 - (9.6 + 64 - 228.6 + 122)$$

$$= 29.8\ dB = 955\ K$$

$$T_s = T_A + T_L + LT_R = T_A + (LF - 1)\,290$$

$955\ K = T_A + (LF - 1)\,290$ $665\ K = (LF - 1)\,290$

$LF = \frac{665}{290} + 1 = 3.293$ $F = \frac{3.293}{L} = \frac{3.293}{1.259}$

$$= 2.615 = 4.2\ dB$$

5.26 $\quad EIRP = \frac{E_b}{N_0} + R + k + L_s + L_0 - G/T$ (in dB)

For DPSK: $P_B = \frac{1}{2}\exp(-E_b/N_0)$

$$2P_B = e^{-E_b/N_0}$$

$$\frac{E_b}{N_0} = -\log_e(2\times10^{-7}) = 15.4 \approx 12\,dB$$

$$L_s = \left(\frac{4\pi d}{\lambda}\right)^2 = \left(\frac{4\pi\times10^4}{3\times10^8/3\times10^9}\right)^2 \doteq 1.58\times10^{12}$$

$$\approx 122\,dB$$

$$EIRP = 12 + 60 - 228.6 + 122 + 30 + L_0 \quad (\text{in dB})$$

$$EIRP = -4.6 + L_0$$

EIRP (dBW)	Watts or mW	Fading Loss (dB)
10	10 Watts	14.6
0	1 Watt	4.6
−3	500 mW	1.6
−4.6	347 mW	0
−5.2	300 mW	−0.6

No, it is not possible to meet the system specifications of 20 dB fading loss with an EIRP less than 10dBW.

5.27

From Problem 5.26, the minimum EIRP corresponding to a 0 dB fading loss is − 4.6 dBW or 347 mW.

$$EIRP = P_t G_t = 347 \text{ mW}$$

$$G_t = \frac{4\pi A_e}{\lambda^2} = \frac{4\pi \times 0.0025}{(3\times10^8/3\times10^9)^2}$$

$$= \frac{0.0314}{0.01} = 3.14$$

where $25 \text{ cm}^2 = 0.0025 \text{ m}^2$

$$P_t = \frac{EIRP}{G_t} = \frac{347 \text{ mW}}{3.14}$$

$$P_t = 110.5 \text{ mW}$$

Chapter 6

6.1 $(n,k) = (8,7)$

$$P_{md} = \binom{8}{2} p^2 (1-p)^6 + \binom{8}{4} p^4 (1-p)^4 + \binom{8}{6} p^6 (1-p)^2 + \binom{8}{8} p^8$$

$$P_{md} = 28(10^{-2})^2(1-10^{-2})^6 + 70(10^{-2})^4(1-10^{-2})^4 + 28(10^{-2})^6(1-10^{-2})^2 + (10^{-2})^8 = 2.6 \times 10^{-3}$$

6.2 $$P_M = \sum_{k=3}^{24} \binom{24}{k} p^k (1-p)^{24-k}$$

$$\cong \binom{24}{3} (10^{-3})^3 (1-10^{-3})^{21} = 1.98 \times 10^{-6}$$

6.3 (a) $P_m^u = 1 - (1-10^{-3})^{92} = 8.8 \times 10^{-2}$

(b) $$P_M^c = \sum_{k=4}^{127} \binom{127}{k} p^k (1-p)^{127-k}$$

$$\cong \binom{127}{4} (10^{-3})^4 (1-10^{-3})^{123}$$

$$= 9.14 \times 10^{-6}$$

6.4 $p_u = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) = Q\left(\sqrt{2\times 10}\right) = Q(4.47)$

$$\cong \frac{1}{x\sqrt{2\pi}} e^{-x^2/2} = \frac{1}{4.47\sqrt{2\pi}} e^{-10} = 4.05\times 10^{-6}$$

$$P_M^u = 1-(1-4.05\times 10^{-6})^{12} = 4.86\times 10^{-5}$$

For the (24, 12) code, the code rate is 1/2. Thus, the data rate is double the uncoded rate, or the E_c/N_0 is 3 dB less than the $\frac{E_b}{N_0}$.

$$E_c/N_0 = 7\text{ dB} = 5.01$$

$$p_c = Q\left(\sqrt{\frac{2E_c}{N_0}}\right) = Q\left(\sqrt{2\times 5.01}\right) = Q(3.16)$$

Using Table B.1, $p_c = 0.0008$

$$P_M^c = \sum_{k=3}^{24}\binom{24}{k} p^k (1-p)^{24-k} \cong \binom{24}{3}(.0008)^3(1-0.0008)^{21}$$

$P_M^c \cong 1.02\times 10^{-6}$ $\quad$ PERFORMANCE IMPROVEMENT $= \frac{4.86\times 10^{-5}}{1.02\times 10^{-6}} = 47.6$

6.5 (a) Noncoherent BFSK with $\frac{E_b}{N_0} = 14\text{dB} = 25.12$

$$p_u = \frac{1}{2} e^{-\frac{1}{2}E_b/N_0} = \frac{1}{2} e^{-\frac{25.12}{2}} = 1.76\times 10^{-6}$$

$$P_M^u = 1-(1-1.76\times 10^{-6})^{12} = 2.11\times 10^{-5}$$

6.5 (a) cont'd. Rate ½ coding.

Thus $\frac{E_c}{N_0} = 11 dB = 12.59$

$$p_c = \frac{1}{2} e^{-\frac{1}{2} E_c/N_0} = \frac{1}{2} e^{-\frac{12.59}{2}} = 9.23 \times 10^{-4}$$

$$P_M^c \cong \binom{24}{3} (9.23 \times 10^{-4})^3 (1 - 9.23 \times 10^{-4})^{21} = 1.56 \times 10^{-6}$$

PERFORMANCE IMPROVEMENT $= \frac{2.11 \times 10^{-5}}{1.56 \times 10^{-6}} = 13.5$

(b) $E_b/N_0 = 10 dB = 10$

$$p_u = \frac{1}{2} e^{-\frac{1}{2} E_b/N_0} = \frac{1}{2} e^{-5} = 3.36 \times 10^{-3}$$

$$P_M^u = 1 - (1 - 3.36 \times 10^{-3})^{12} = 3.96 \times 10^{-2}$$

Rate ½ code $E_c/N_0 = 7 dB = 5.01$

$$p_c = \frac{1}{2} e^{-\frac{1}{2} E_c/N_0} = \frac{1}{2} e^{-2.5} = 4.1 \times 10^{-2}$$

$$P_M^c \cong \binom{24}{3} (4.1 \times 10^{-2})^3 (1 - 4.1 \times 10^{-2})^{21}$$

$$= 5.7 \times 10^{-2}$$

There is a performance degradation $= \frac{5.7 \times 10^{-2}}{3.96 \times 10^{-2}} = 1.4$

due to the fact that the E_b/N_0 is not large enough for the code to exhibit its coding gain properties. At this value of E_b/N_0 the code digits are just "excess baggage."

6.6 A decoding error will be made if three or more of the repetitions are received in error. Thus,

$$P_B = \sum_{j=3}^{5} \binom{5}{j} p^j (1-p)^{5-j} \cong \binom{5}{3} (10^{-3})^3 (1-10^{-3})^2 \cong 10^{-8}$$

6.7 $d_{min} = 11$

error correcting: $t = \frac{d_{min}-1}{2} = 5$

error detecting: $m = d_{min} - 1 = 10$

erasure correcting: $\rho = d_{min} - 1 = 10$

6.8 (a) $\underline{U} = \underline{m}\, G$

messages	code vectors
0000	0000000
0001	1100001
0010	0110010
0011	1010011
0100	1010100
0101	0110101
0110	1100110
0111	0000111
1000	1111000
1001	0011001
1010	1001010
1011	0101011
1100	0101100
1101	1001101
1110	0011110
1111	1111111

(b) $H = \left[I_{n-k} \,\vdots\, P^T \right] = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$

(c) $\underline{S} = \underline{r} H^T = [1101101] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} = [010]$

Thus, 1101101 is <u>not</u> a valid codeword

(d) d_{min} = minimum weight = 3

$t = \dfrac{d_{min} - 1}{2} = 1$

(e) $m = d_{min} - 1 = 2$

<u>6.9</u> (a) $G = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$

$H = \left[I_{n-k} \,\vdots\, P^T \right] = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$

(b) $t = \left\lfloor \dfrac{d_{min} - 1}{2} \right\rfloor = \left\lfloor \dfrac{4-1}{2} \right\rfloor = 1$

(c) $\underline{S} = \underline{r} H^T = [10101010]\, H^T = [0011]$ <u>Not</u> a codeword

(d) $\underline{S} = [01011100]\, H^T = [0000]$ <u>yes</u>; it is a codeword.

(b) The generator matrix, here, consists of $k = 2$ (linearly independent) basis vectors.

$$G = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Any two of the three nonzero vectors would do equally well in this case.

(c) $H = \left[I_{m-k} \mid P^T \right] = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$

(d)

0000	1001	1110	0111
0001	1000	1111	0110
0010	1011	1100	0101
0100	1101	1010	0011

(e) $d_{min} = 2$; Therefore $t = 0$ meaning that, although some of the single-error patterns are correctable, they are not all correctable.

$m = d_{min} - 1 = 1$

(f) $\underline{S} = \underline{e} H^T$

coset leader	syndrome
0000	00
0001	10
0010	11
0100	01

* Note again, that this is a design problem with more than one solution.

6.10

(a)

$$G = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(b)

$$H = \left[I_{n-k} \mid P^T \right] = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

(c) $n = 9$, $k = 5$, $d_{min} = 3$ (from the minimum weight vector, out of the set of 32 code vectors).

6.11 (a) Constraints

1. systematic form
2. maximize d_{min}, hence maximize the minimum weight
3. all-zeros vector must be a member of the codeword set.
4. the codeword set must exhibit closure.

$2^k = 2^2 = 4$ messages

∴ there are 4 codewords

message	chosen codewords
00	00000
01	01101
10	10110
11	11011

Check for closure

```
 01101
 10110
 -----
 11011 ✓
```

Are there other solutions? Yes. Here is another codeword set.

00000
10101
11010
01111

6.11 (cont'd.)

(b) Generator Matrix for the Chosen codeword set

$$G = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

(c) Parity Check Matrix

$$H = \left[I_{n-k} \,\vdots\, P_T \right]$$

$$= \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

(d) Standard Array

00000	01101	10110	11011
00001	01100	10111	11010
00010	01111	10100	11001
00100	01001	10010	11111
01000	00101	11110	10011
10000	11101	00110	01011
01010	00111	11100	10001
11000	10101	01110	00011

(e) Minimum weight of codewords $= 3$

$d_{min} = 3$

Error correcting $t = \frac{d_{min} - 1}{2} = 1$

Error detecting $m = d_{min} - 1 = 2$

(f) Syndrome table for correcting the single errors: $S = e \cdot H^T = e \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

Error Pattern	Syndrome
00001	011
00010	101
00100	001
01000	010
10000	100

$\boxed{00000}$
00001
00010
00011
00100
00101
00110
00111
01000
01001
01010
01011
01100
$\boxed{01101}$
01110
01111
10000
10001
10010
10011
10100
10101
$\boxed{10110}$
10111
11000
11001
11010
$\boxed{11011}$
11100
11101
11110
11111

6.12

00000	11111
00001	11110
00010	11101
00100	11010
01000	10111
10000	01111
00011	11100
00101	11010
01001	10110
10001	01110
00110	11001
01010	10101
10010	01101
01100	10011
10100	01011
11000	00111

The error patterns (coset leaders) in the left hand column comprise all 1-error and 2-error patterns, and nothing more. Thus, the code is a perfect code.

6.13

messages	codewords
0	000
1	111

$d_{min} = 3$

$t = \frac{d_{min} - 1}{2} = 1$

Standard array:

000	111
001	110
010	101
100	011

6.14

(7, 3) code: n-tuples $= 2^7 = 128$; codewords $= 2^3 = 8$; Standard array $= 16 \times 8$ array. Thus, the 16 coset leaders allow for the

correction of all single-error patterns and eight of the double-error patterns. However, the possible number of double-error patterns are $\binom{7}{2} = 21$. Thus, a (7,3) code is not a perfect code.

(7,4) code: n-tuples $= 2^7 = 128$; codewords $= 2^4 = 16$; Standard array $= 8 \times 16$. Thus, the 8 coset leaders allow for the correction of all single-error patterns, and nothing more. Thus, a (7,4) code is a perfect code.

(15,11) code: n-tuples $= 2^{15} = 32{,}768$; codewords $= 2^{11} = 2048$; Standard array $= 16 \times 2048$. Thus, the 16 coset leaders allow for the correction of all single-error patterns, and nothing more. Thus, a (15,11) code is a perfect code.

6.15 (a) $\mathbf{H} = \left[\mathbf{I}_{n-k} \,\vdots\, \mathbf{P}^T \right]$

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

(b)

000000000000000
000000000000001
000000000000010
000000000000100
000000000001000
000000000010000
000000000100000
000000001000000
000000010000000
000000100000000
000001000000000
000010000000000
000100000000000
001000000000000
010000000000000
100000000000000

A (15, 11) code is a perfect code. The 16 coset leaders allow for the correction of all single-error patterns, and no double-error patterns.

(c) $\underline{S} = \underline{r}\,H^T = [011111001011011]\,H^T = [0110]$

Thus, $\underline{r}$ is not a codeword. The coset leader resulting in the syndrome [0110], is 000000010000000. Therefore, the correct codeword is: 011111011011011.

(d) With the knowledge that $t_{max} = 1$, and the examination of $G = [P \vdots I_k]$, we see that $d_{min} = 3$. Therefore, $\rho = d_{min} - 1 = 2$. If vector $\underline{r}$ = XX1111011011011 is received (where XX stands for 2 erasures), it will be decoded as 011111011011011 since $\underline{r}$ is closest in Hamming distance to this codeword than to any of the other codewords (when comparing the rightmost 13 digits).

6.16 YES. There are $2^k - 1$ nonzero error patterns that will alter a transmitted codeword $\underline{U}_i$ into another codeword $\underline{U}_j$. From Figure 6.11, there are $2^3 - 1 = 7$ nonzero error patterns that cannot be detected. They are seen as the row of nonzero codewords.

Example from Figure 6.11: If codeword 110011 is transmitted, and the error pattern 000111 changes it so that the received vector is $\underline{r}$ = 110011 + 000111 = 110100 (another codeword), then the syndrome $\underline{S} = \underline{r}\,\mathbf{H}^T = 0$.

6.17 Test: Does $x^n + 1 = q(x)\,g(x)$?

(a) $1 + x^3 + x^4$; $n - k = 4$. Then, for $k = 1, 2, 3$, $n = 5, 6, 7$, respectively.

$\underline{n = 5}$: $$\frac{x^5+1}{x^4+x^3+1} = x + 1 + \frac{x^3+x}{x^4+x^3+1} \qquad \underline{\underline{\text{No}}}$$

$\underline{n = 6}$: $$\frac{x^6+1}{x^4+x^3+1} = x^2 + x + 1 + \frac{x^3+x^2+x}{x^4+x^3+1} \qquad \underline{\underline{\text{No}}}$$

$\underline{n = 7}$: $$\frac{x^7+1}{x^4+x^3+1} = x^3 + x^2 + x + 1 + \frac{x^2+x}{x^4+x^3+1} \qquad \underline{\underline{\text{No}}}$$

(b) $1 + x^2 + x^4$; $n - k = 4$. Then, for $k = 1, 2, 3$, $n = 5, 6, 7$, respectively.

$n=5$: $\dfrac{x^5+1}{x^4+x^2+1} = x+1 + \dfrac{x^3+x+1}{x^4+x^2+1}$ No

$n=6$: $\dfrac{x^6+1}{x^4+x^2+1} = x^2+1$ YES

$n=6, \quad n-k=4$

code that can be generated $\Rightarrow (n,k) = (6,2)$

$n=7$: $\dfrac{x^7+1}{x^4+x^2+1} = x^3+x + \dfrac{x+1}{x^4+x^2+1}$ No

(c) $1+x+x^3+x^4$ $n-k=4$. Then for $k=1,2,3$, $n=5,6,7$, respectively.

$n=5$: $\dfrac{x^5+1}{x^4+x^3+x+1} = x+1 + \dfrac{x^3+x^2}{x^4+x^3+x+1}$ No

$n=6$: $\dfrac{x^6+1}{x^4+x^3+x+1} = x^2+x+1$ YES

$n=6, \quad n-k=4$

code that can be generated $\Rightarrow (n,k) = (6,2)$

$n=7$: $\dfrac{x^7+1}{x^4+x^3+x+1} = x^3+x^2+x + \dfrac{x+1}{x^4+x^3+x+1}$ No

(d) $1+x+x^2+x^4$ $n-k=4$. Then for $k=1,2,3$, $n=5,6,7$, respectively.

$n=5$: $\dfrac{x^5+1}{x^4+x^2+x+1} = x + \dfrac{x^3+x^2+x+1}{x^4+x^2+x+1}$ No

$n=6$: $\dfrac{x^6+1}{x^4+x^2+x+1} = x^2+1 + \dfrac{x^3+x+1}{x^4+x^2+x+1}$ No

$n=7$: $\dfrac{x^7+1}{x^4+x^2+x+1} = x^3+x+1$ YES

$n=7, \quad n-k=4$

code that can be generated $\Rightarrow (n, k) = (7, 3)$

(e) $1+x^3+x^5 \quad n-k=5$. Then, for $k=1, 2$,
$n = 6, 7$, respectively.

$n=6$: $\dfrac{x^6+1}{x^5+x^3+1} = x + \dfrac{x^4+x+1}{x^5+x^3+1}$ No

$n=7$: $\dfrac{x^7+1}{x^5+x^3+1} = x^2+1 + \dfrac{x^3+x^2}{x^5+x^3+1}$ No

6.18 $\quad \underline{m} = 101$; $m(X) = 1+X^2$ $\boxed{k=3}$

$n-k=4$; $n=7$; $(n,k) = (7,3)$

$g(X) = 1+X+X^2+X^4$

$X^{n-k}m(X) = X^4(1+X^2) = X^4+X^6$

$X^{n-k}m(X) = q(x)\,g(x) + \underbrace{r(x)}_{\text{REMAINDER}}$

$$
\begin{array}{r}
X^2 \\
X^4 + X^2 + X + 1 \overline{\big) X^6 + X^4} \\
\underline{X^6 + X^4 + X^3 + X^2} \\
X^3 + X^2
\end{array}
\quad \text{REMAINDER}
$$

$$\underbrace{\underbrace{r(X)}_{\text{PARITY}} + \underbrace{X^{n-k} m(X)}_{\text{MESSAGE}}}_{\text{CODEWORD}} = X^2 + X^3 + X^4 + X^6$$

$$= \underbrace{0011}_{\text{PARITY}}\ \underbrace{101}_{\text{MESSAGE}}$$

6.19

SWITCH

m(X) INPUT

OUTPUT

INPUT QUEUE	SHIFT #	REGISTER CONTENTS
10101	0	000
1010	1	111
101	2	100
10	3	101
1	4	101
–	5	010 (PARITY)

Therefore, the final codeword is: 01010101

6.20 $P_r/N_0 = 48$ dBW ; $R_c = 40$ dB-bits/s

$$E_c/N_0 = (1/R_c)\, P_r/N_0 = 48 - 40 = 8 \text{ dB} = 6.31$$

$$p_c = \frac{1}{2}\exp(-E_c/N_0) = \frac{1}{2}\exp(-6.31)$$
$$= 9.09 \times 10^{-4}$$

$$P_M \cong \binom{7}{2} p_c^2 (1-p_c)^5 = 1.73 \times 10^{-5}$$

__YES__. $\frac{P_r}{N_0} = 48$ dBW is sufficient to provide a $P_M \leq 10^{-3}$.

__6.21__ (a) $(n,k) = (15,5)$; $n-k = 10$

SWITCH

INPUT

OUTPUT

(b) $m(X) = 1 + X^2 + X^4$

$$X^{n-k} m(X) = X^{10}(1 + X^2 + X^4) = X^{10} + X^{12} + X^{14}$$

$$= q(X)\, g(X) + r(X)$$

$$\begin{array}{r} X^4 + 1 \\ X^{10}+X^8+X^5+X^2+X+1 \;\overline{)\; X^{14}+X^{12}+X^{10}} \\ X^{14}+X^{12}+X^9+X^6+X^5+X^4 \\ \hline X^{10}+X^9+X^6+X^5+X^4 \\ X^{10}+X^8+X^5+X^2+X+1 \\ \hline X^9+X^8+X^6+X^4+X^2+X+1 \end{array}$$

REMAINDER $\Rightarrow X^9+X^8+X^6+X^4+X^2+X+1$

$$\underbrace{r(X) + X^{n-k} m(X)}_{\text{CODEWORD}} = 1 + X + X^2 + X^4 + X^6 + X^8 + X^9 + X^{10} + X^{12} + X^{14}$$

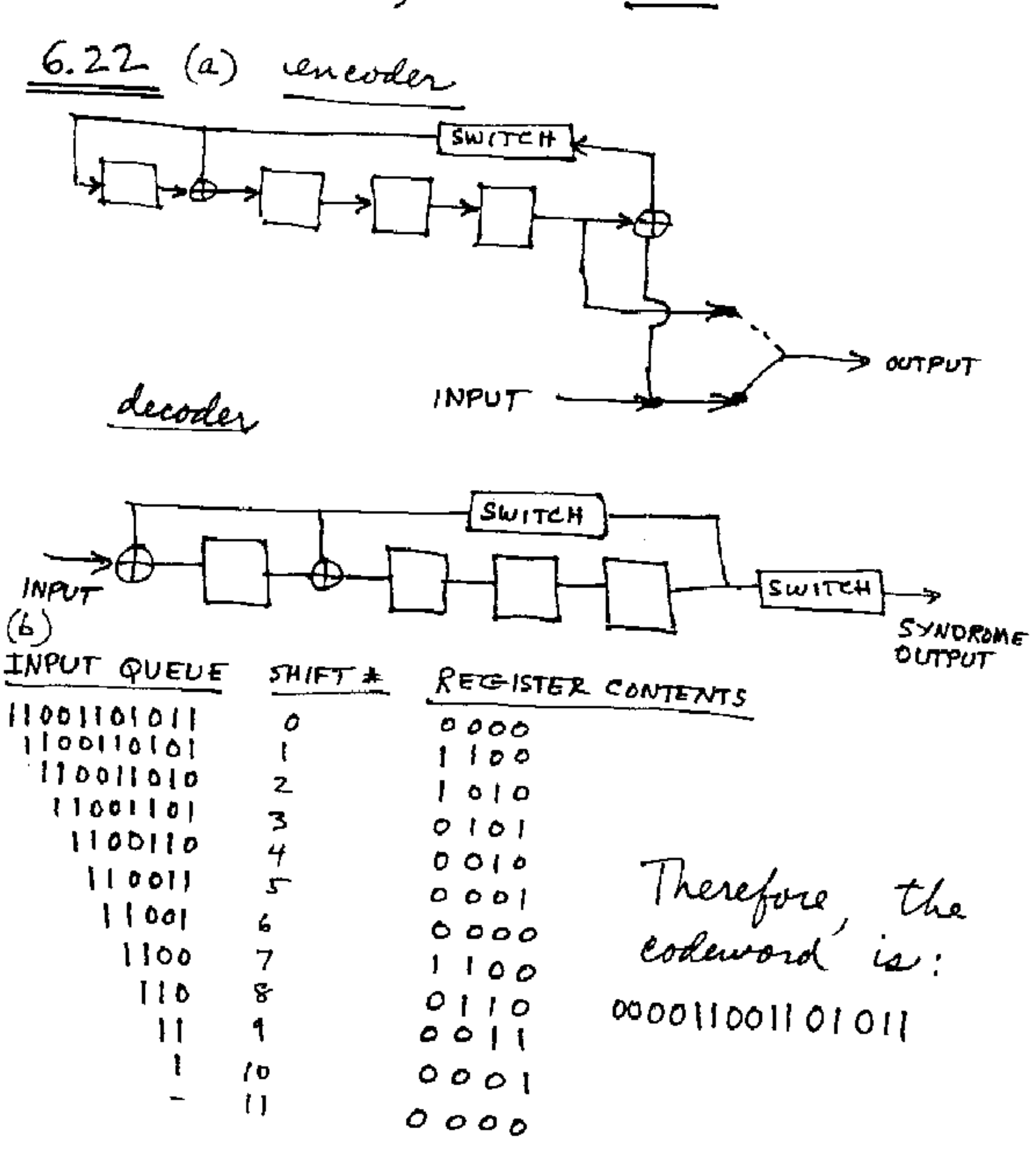
$$= \underbrace{1\,1\,1\,0\,1\,0\,1\,0\,1\,1}_{\text{PARITY}}\ \underbrace{1\,0\,1\,0\,1}_{\text{MESSAGE}}$$

(c) TEST: Divide $V(X)$ by $g(X)$. It is a codeword if $r(X) = 0$.

$$\frac{X^{14}+X^8+X^6+X^4+1}{X^{10}+X^8+X^5+X^2+X+1} = X^4+X^2+1+\frac{X^9+X^7+X^4+X^3+X}{X^{10}+X^8+X^5+X^2+X+1}$$

Since $r(X) \neq 0$, $V(X)$ is <u>not</u> a codeword.

6.22 (a) encoder



(b)

INPUT QUEUE	SHIFT #	REGISTER CONTENTS
11001101011	0	0000
1100110101	1	1100
110011010	2	1010
11001101	3	0101
1100110	4	0010
110011	5	0001
11001	6	0000
1100	7	1100
110	8	0110
11	9	0011
1	10	0001
–	11	0000

Therefore, the codeword is:

000011001101011

(c) decoding procedure:

INPUT QUEUE	SHIFT #	REGISTER CONTENTS
000011001101011	0	0000
00001100110101	1	1000
0000110011010	2	1100
000011001101	3	0110
00001100110	4	1011
0000110011	5	1001
000011001	6	0000
00001100	7	1000
0000110	8	0100
000011	9	0010
00001	10	1001
0000	11	0000
000	12	0000
00	13	0000
0	14	0000
-	15	0000

6.23

The (15, 11) code introduces less redundancy, so it has less error correcting capability.

The (15, 11) code, because of lower redundancy requires less bandwidth. Trade-off is required power versus required bandwidth.

6.24 (a) The (63, 36) code can correct only five errors, but the errors can occur in any pattern among the 63 bits. The (7, 4) code can correct up to nine errors, but only if they are

distributed one error per codeword block (which is unlikely). Therefore, the (7,4) code is not nearly as powerful as the number of correctable errors in nine blocks implies.

(b) The (63,36) code can correct all error patterns containing 5 or less bit errors. The (7, 4) code requires that there is ≤ 1 bit error per block, in order for the decoding to be successful. Given 1 bit error in any block, the probability that the second bit error is not in the same block is 8/9. Given those two bit errors in separate blocks, the probability of a third bit error not in those blocks is 7/9. The probability of 5 bit errors in separate blocks is: $\dfrac{8 \times 7 \times 6 \times 5}{9^4}$ $= 0.256$. Thus, the (7,4) code will successfully decode such errors, only one quarter of the time.

6.25 (a) $P_M^u = 1-(1-p_u)^{36} \cong 36\, p_u = 10^{-3}$

$$p_u \cong \frac{10^{-3}}{36} = 2.87 \times 10^{-5}$$

$p_u = \frac{1}{2} \exp\left(-\frac{1}{2} E_b/N_0\right)$; $\frac{E_b}{N_0} = -2 \ln(2\, p_u)$

$E_b/N_0 = 19.6 = 12.92$ dB without coding

(b) Use of a (127, 36) code with $d_{min} = 31$ can correct $t_{max} = 15$ errors.

$$P_M^c \cong \binom{127}{16} p_c^{16} (1-p_c)^{111} = 10^{-3}$$

Solving iteratively for p_c yields $p_c \cong 0.0546$

$$p_c = \frac{1}{2} \exp\left(-\frac{1}{2} E_c/N_0\right) = 0.0546$$

$$E_c/N_0 = -2 \ln(2\, p_c) = 4.43 = 6.46 \text{ dB}$$

$$E_b/N_0 = \frac{127}{36} \frac{E_c}{N_0} = 15.63 = 11.94 \text{ dB}$$

$\therefore$ Coding gain $= 12.92 - 11.94 = 0.98$ dB

6.26 (a)

Data rate, R → (127, 64) code → Coded data rate R_c → 16-ary PSK Modulation → Symbol rate, R_s

P_B ← Decode ← P_E, P_b ← Demodulation ← $E_b/N_0 = 10$ dB

where P_E is the symbol-error probability, P_b is the coded-bit-error probability, and P_B is the information-bit (decoded-bit) error probability.

$\frac{E_b}{N_0}$ received $= 10\ dB = 10$

$$\frac{E_c}{N_0} = 10 \times \frac{64}{127} = 5.039 = 7.024\ dB$$

since the information bits are rate $64/127$ encoded.

$$\frac{E_s}{N_0} = (\log_2 M)\frac{E_c}{N_0} = \frac{4E_c}{N_0} = 20.16 = 13.04\ dB$$

From Equation (4.105)

$$P_E = 2Q\left[\left(\sqrt{\frac{2E_s}{N_0}}\right)\left(\sin\frac{\pi}{M}\right)\right] = 2Q(1.24)$$

$$= 2 \times 0.1066 = 0.213$$

$$P_b \cong P_E/k = 0.213/4 = 0.0533$$

Since the (127, 64) BCH code corrects all error patterns up to $t = 10$, using Equation (6.46) the decoded-bit-error probability, P_B, is:

$$P_B \cong \frac{1}{n}\sum_{j=t+1}^{n} j\binom{n}{j} P_b^j (1-P_b)^{n-j}$$

$$\cong \frac{11}{127}\binom{127}{11}P_b^{11}(1-P_b)^{116} + \frac{12}{127}\binom{127}{12}P_b^{12}(1-P_b)^{115} + \cdots$$

$$\cong 7 \times 10^{-3}$$

which is the information-bit-error probability or the decoded-bit error probability.

(b) For a decoded-bit error probability of 7×10^{-3} the uncoded or channel-bit error probability is as in part a) $P_b = 0.0533$

prob. of symbol error } $P_E = \frac{2^k - 1}{2^{k-1}} P_b = \frac{15}{8} \times 0.0533$

$$= 0.0999$$

$$P_E \le (M-1)\, Q\left(\sqrt{E_s/N_0}\right) = 0.0999$$

$$Q\left(\sqrt{E_s/N_0}\right) = \frac{0.0999}{15} = 0.00666$$

Using Table 4.1,

$$\sqrt{E_s/N_0} = 2.48$$

$$E_s/N_0 = 6.15$$

$$E_c/N_0 = \left(\frac{1}{k}\right)\frac{E_s}{N_0} = \frac{6.15}{4} = 1.5376$$

$$E_b/N_0 = \frac{127}{64}\,\frac{E_c}{N_0} = 3.05$$

$$= 4.84 \text{ dB}$$

Comparison with part (a) agrees with our intuition. For a given error performance the E_b/N_0 for 16-ary FSK should be less than that for 16-ary PSK, as can be verified by comparing Figure 4.28 with Figure 4.29.

6.27 (a) Let P_M be the probability that a word or message is in error.

$$P_M = 1-(1-p)^{7\times 6} = 1-(1-10^{-3})^{42} = \underline{4.1\times 10^{-2}}$$

(b) Let P_c be the probability that a word is correct, and let P_{cc} be the probability that a character within the word is correct.

$$P_M = 1-P_c \;; \quad P_c = P_{cc}^6$$

$$P_{cc} = \underbrace{(1-p)^{7\times 3}}_{\text{prob that each of the 3 repetitions are decoded correctly}} + \underbrace{\binom{3}{2}(1-p)^{7\times 2}\left[1-(1-p)^7\right]}_{\text{probability that 2 of the 3 repetitions are decoded correctly and 1 of the repetitions is decoded incorrectly.}}$$

$$P_M = 1-P_c = 1-P_{cc}^6$$

$$= 1-\left\{(1-10^{-3})^{21} + 3(1-10^{-3})^{14}\left[1-(1-10^{-3})^7\right]\right\}^6$$

$$= \underline{8.7\times 10^{-4}}$$

(c) $P_M \cong \binom{126}{5} p^{15} (1-p)^{111}$

$= \dfrac{126!}{15!\ 111!} (10^{-3})^{15} (1-10^{-3})^{111} = \underline{9.2 \times 10^{-27}}$

(d) Repeat of (a): $\dfrac{E_b}{N_0} = 12\text{dB} = 15.85$

channel error prob $p = \frac{1}{2} e^{-\frac{15.85}{2}} = 1.8 \times 10^{-4}$

$P_M = 1 - (1 - 1.8 \times 10^{-4})^{42} = \underline{7.5 \times 10^{-3}}$

Repeat of (b): Coding is rate ⅓ since 200% redundancy is introduced.

Therefore $\dfrac{E_c}{N_0} = \dfrac{E_b}{3 N_0} = \dfrac{15.85}{3}$

$p = \frac{1}{2} e^{-\frac{15.85}{6}} = \underline{3.56 \times 10^{-2}}$

$P_M = 1 - \left\{ (1-3.56\times10^{-2})^{21} + 3(1-3.56\times10^{-2})^{14}\left[1-(1-3.56\times10^{-2})^{7}\right] \right\}^{6}$

$= 5.6 \times 10^{-1}$

Repeat of (c): Rate ⅓ code, $p = 3.56 \times 10^{-2}$

$P_M \cong \binom{126}{15} (3.56 \times 10^{-2})^{15} (1 - 3.56 \times 10^{-2})^{111}$

$= \underline{3.4 \times 10^{-5}}$

(e) Operating a communication system with the symbol error probability fixed regardless of the message redundancy implies that the E_b/N_0 must be increased for increased redundancy. Under such conditions we see that the repetition code provides about 16 dB error performance improvement over the uncoded case, and the BCH code provides an enormous improvement over the other two cases. A more realistic comparison of coding capability is one where the system operates with a fixed E_b/N_0. Here we see that the repetition code results in nearly 35 dB of degraded error performance, while the BCH code offers about 7 dB of coding gain compared to the uncoded case. Therefore, a repetition code offers improvement when the received E_b/N_0 is increased (i.e., by increasing transmission power or increasing transmission duration and thus delay). Otherwise, the repetition code causes degradation.

6.28 For the coded case, the bound in Equation (6.7) can be used:

$$P_B(M) \le \frac{M}{2} Q\left(\sqrt{\frac{E_s}{N_0}}\right)$$

Since $k = 5$ bits, $M = 2^k = 32$. Using a $P_B = 10^{-5}$ reference (coding gain must be associated with a particular P_B), we write

$$10^{-5} = \frac{32}{2} Q\left(\sqrt{\frac{E_s}{N_0}}\right) = 16\, Q\left(\sqrt{\frac{k E_b}{N_0}}\right)$$

Let us use the approximation for $Q(\cdot)$, given in Equation (3.44), and then solve for E_b/N_0 by trial-and-error.

$$P_B = \frac{1}{x\sqrt{2\pi}} e^{-x^2/2} \quad \text{where } x = \sqrt{\frac{E_s}{N_0}} = \sqrt{\frac{k E_b}{N_0}}$$

Thus, $10^{-5} = \dfrac{16}{x\sqrt{2\pi}} e^{-x^2/2}$

Solving for x yields $x = 4.854$

Therefore, $\sqrt{\dfrac{5E_b}{N_0}} = 4.854$

$$\frac{E_b}{N_0} = 4.712 \text{ (or approx. 6.7 dB)}$$

Since uncoded BPSK for a $P_B = 10^{-5}$ (with perfect synchronization) requires an $E_b/N_0 = 9.6$ dB, then the coding gain is:

$$G(\text{dB}) = \left(\frac{E_b}{N_0}\right)_u (\text{dB}) - \left(\frac{E_b}{N_0}\right)_c (\text{dB})$$

$$= 9.6 - 6.7 = 2.9 \text{ dB}$$

$\underline{6.29}$ We verify the generator matrix by generating the 4 codewords as:

$\underline{u} = \underline{m}[G]$

$\underline{u}_1 = 00 \begin{bmatrix} 0 0 1 1 1 1 1 0 \\ 1 1 1 1 0 0 0 1 \end{bmatrix} = 00000000$

$\underline{u}_2 = 01\ [G] = 11110001$

$\underline{u}_3 = 10\ [G] = 00111110$ ✓

$\underline{u}_4 = 11\ [G] = 11001111$

$H = [I_{n-k} \vdots P^T] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$ ✓

Parity check matrix constructed from [G]

Syndrome $\underline{S} = \underline{e} \cdot H^T$

$= \underline{e} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}$

$\underline{S}_1 = [00000000] \cdot H^T = 000000$

$\underline{S}_2 = [00000001] \cdot H^T = 111100$

$\underline{S}_3 = [00000010] \cdot H^T = 001111$ ✓

$\underline{S}_4 = [00000100] \cdot H^T = 000001$

$\underline{S}_5 = [00001000] \cdot H^T = 000010$

6.29 (cont'd.)

$\underline{S}_6 = [00010000] \cdot H^T = 000100$

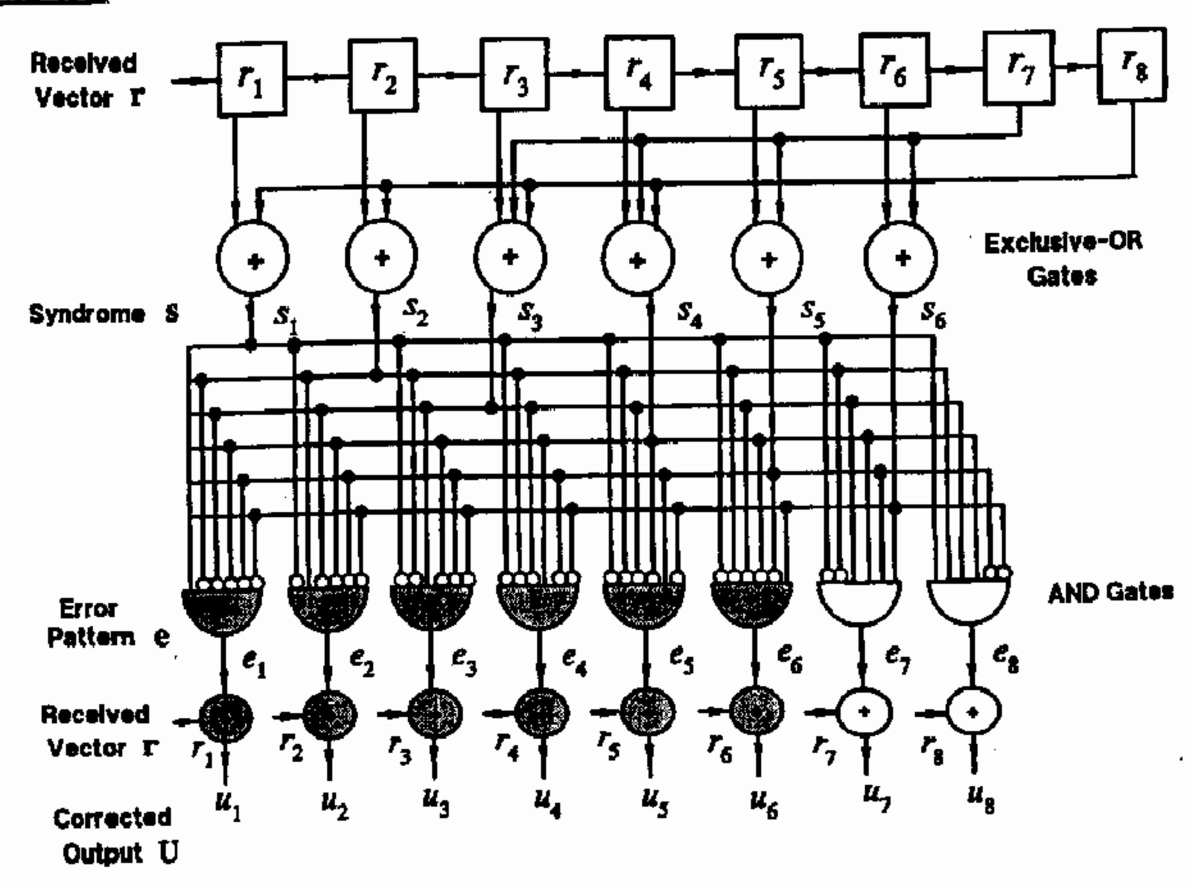
$\underline{S}_7 = [00100000] \cdot H^T = 001000$

$\underline{S}_8 = [01000000] \cdot H^T = 010000$ ✓

$\underline{S}_9 = [10000000] \cdot H^T = 100000$

$\underline{S}_{10} = [00000011] \cdot H^T = 110011$

6.30



For codes in systematic form, the decoder only delivers the message bits (u_7 and u_8). Hence the gates shown with shading, can be eliminated.

6.31

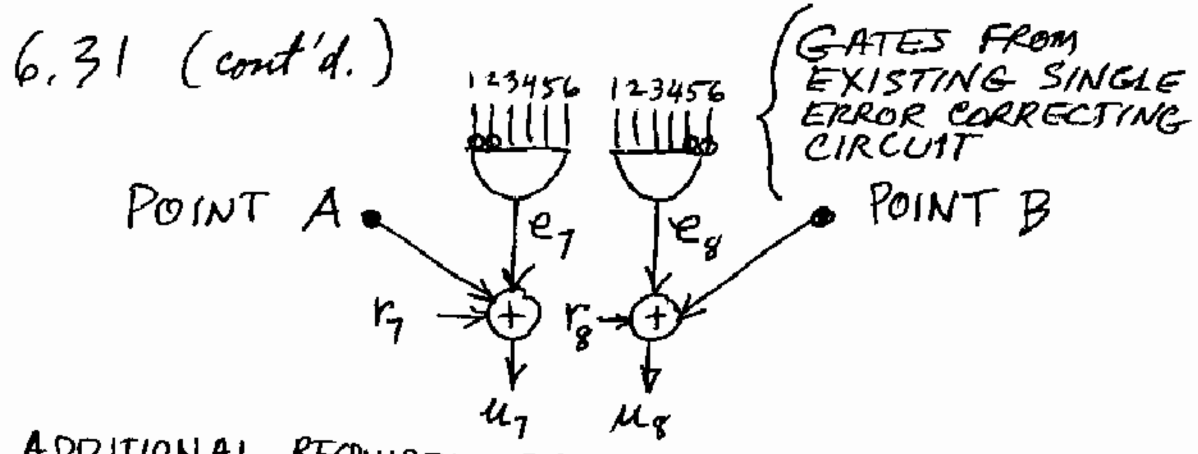
To correct all single and double errors with the (8,2) code, whose standard array is shown in Figure 6.15, we need the circuitry shown in the solution to Problem 6.30 (for correcting single errors) plus additional gates as follows:

Assume that the code is in systematic form, so that the decoder need only deliver the rightmost 2 bits (data) of each 8-bit codeword.

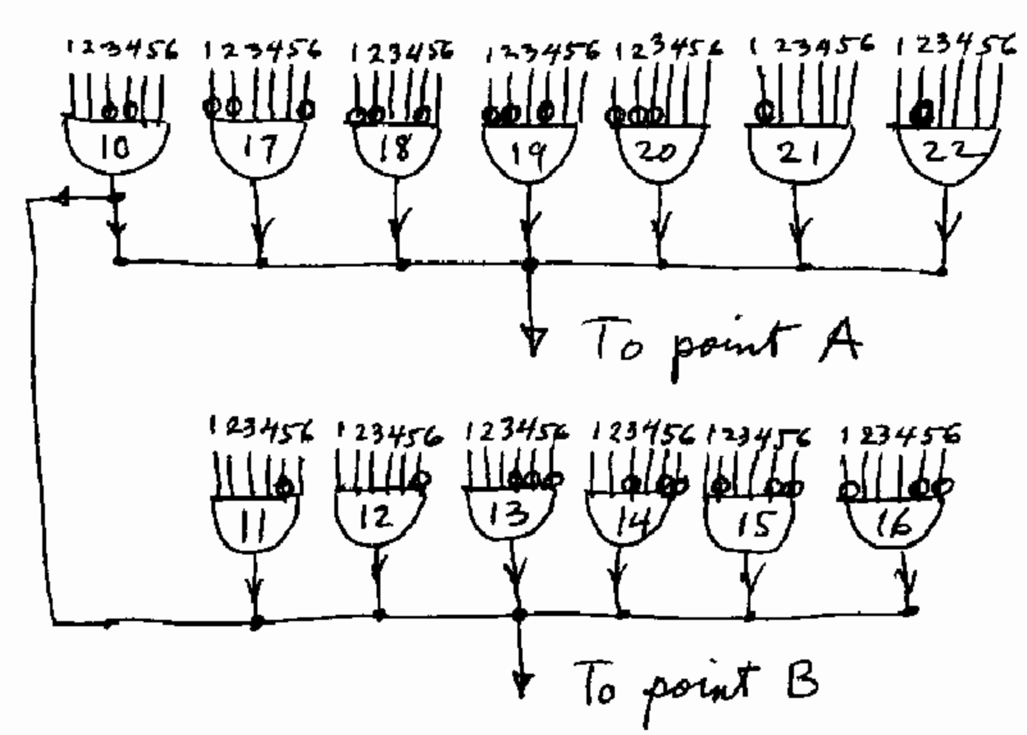
To correct double errors that affects the data means that the circuitry must additionally deliver the proper error pattern whenever the error corresponds to one of the data errors in rows 10-22 of the standard array.

One possible circuit implementation to accomplish this would consist of the circuit shown in the solution to Problem 6.30, with additional gates as shown:

6.31 (cont'd.)



ADDITIONAL REQUIRED GATES:



Where ▽ represents an AND gate, the wires labeled 1, ..., 6 are connected to syndrome digits $s_1, \ldots, s_6$, and a small circle on a wire means "the compliment of". The number in each AND gate represents the error pattern (numbers 10-22) being mitigated by the output of that gate.

6.31 (cont'd.)

Still more gates will be needed to test for those cases where the syndromes are <u>nonzero</u> but no correction is designed to take place. This is useful to perform error detection for any of the syndromes numbered 38 through 64 (in this example).

<u>6.32</u> BCH codes with $n = 31$

$d_{min} = 2t+1$

(n, k)	t	$n-k$
31, 26	1	5
31, 21	2	10
31, 16	3	15
31, 11	5	20
31, 6	7	25

<u>Hamming Bound</u>

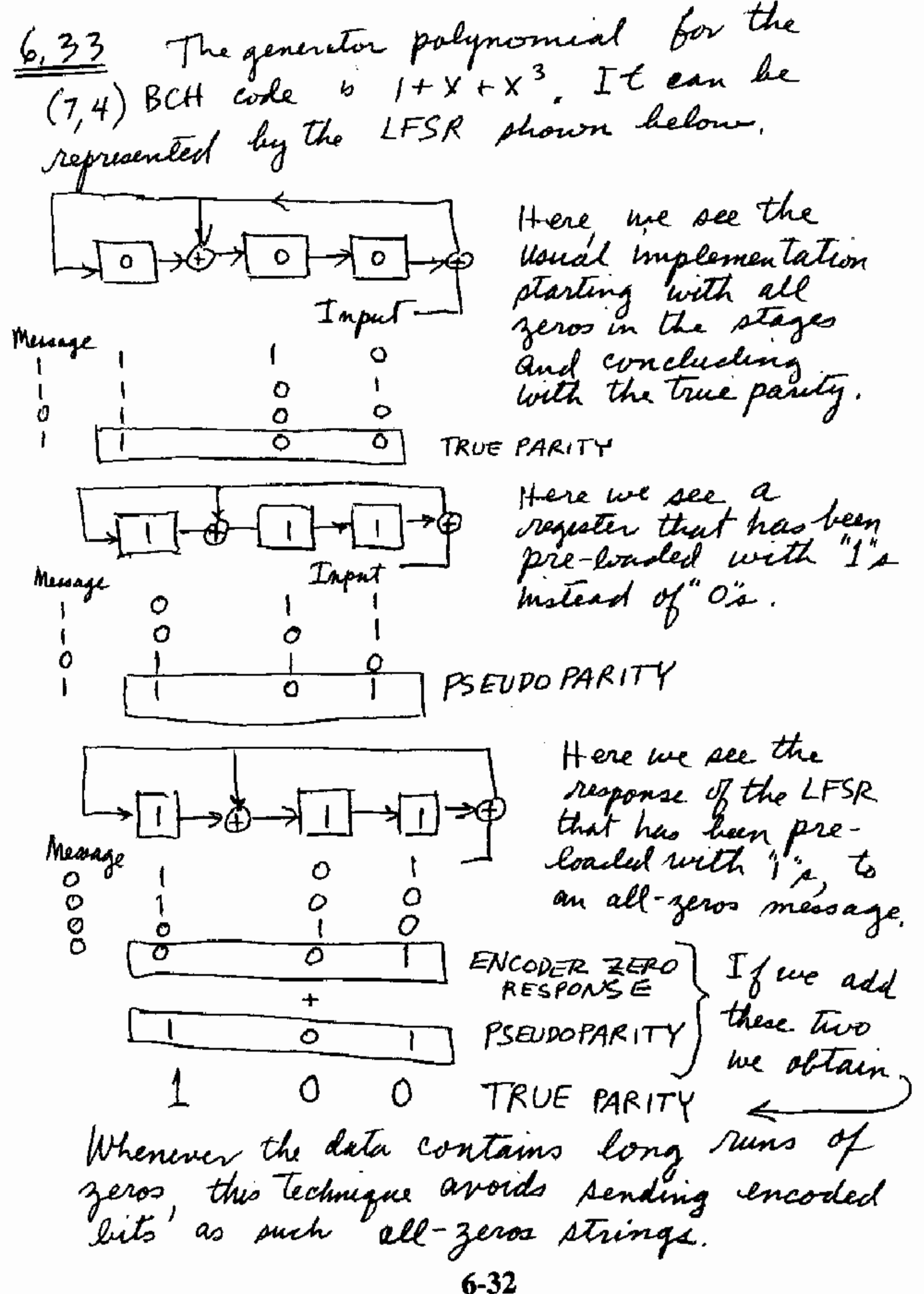
$$2^{n-k} \geq \left[1 + \binom{n}{1} + \cdots \binom{n}{t}\right]$$

<u>Plotkin Bound</u>

$$d_{min} \leq \frac{n \times 2^{k-1}}{2^k - 1}$$

CODE	MEETS HAMMING	MEETS PLOTKIN
31, 26	$32 \geq 32$	$3 \leq 31$
31, 21	$1024 \geq 497$	$5 \leq 31$
31, 16	$32{,}768 \geq 4992$	$7 \leq 31$
31, 11	$1{,}098{,}576 \geq 206{,}368$	$11 \leq 31$
31, 6	$33{,}554{,}432 \geq 3{,}572{,}224$	$15 \leq 31$

6.33 The generator polynomial for the (7,4) BCH code is $1+X+X^3$. It can be represented by the LFSR shown below.



Whenever the data contains long runs of zeros, this technique avoids sending encoded bits as such all-zeros strings.

6.34 (a) $\underline{m} = 1\,1\,0\,1\,1$

$\underline{m}(X) = 1 + X + X^3 + X^4$

$X^{n-k}\,\underline{m}(X) = \underline{q}(X)\,\underline{g}(X) + \underline{p}(X)$

$X^{10}(1 + X + X^3 + X^4) = X^{10} + X^{11} + X^{13} + X^{14}$

where $\underline{g}(X) = 1 + X + X^2 + X^5 + X^8 + X^{10}$

The degree of $\underline{g}(X) = n-k$.

The parity polynomial $p(X)$ is the remainder that results from dividing the upshifted $\underline{m}(X)$ by $\underline{g}(X)$.

$$X^{10}+X^8+X^5+X^2+X+1 \;\big)\; \overset{X^4+X^3+X^2}{X^{14}+X^{13}+X^{11}+X^{10}}$$

$$X^{14}+X^{12}+X^9+X^6+X^5+X^4$$

$$X^{13}+X^{12}+X^{11}+X^{10}+X^9+X^6+X^5+X^4$$

$$X^{13}+X^{11}+X^8+X^5+X^4+X^3$$

$$X^{12}+X^{10}+X^9+X^8+X^6+X^3$$

$$X^{12}+X^{10}+X^7+X^4+X^3+X^2$$

PARITY $X^9+X^8+X^7+X^6+X^4+X^2$

$\underline{U}(X) = \boxed{X^{n-k}\,\underline{m}(X) + \underline{p}(X)} = \underline{q}(X)\,\underline{g}(X)$

$\underline{U}(X) = X^2+X^4+X^6+X^7+X^8+X^9+X^{10}+X^{11}+X^{13}+X^{14}$

or in binary form: $\underline{U} = \underbrace{0010101111}_{\text{parity}}\ \underbrace{11011}_{\text{data}}$

6.34 (a) (cont'd.) Safety check:

$$\frac{\underline{U}(X)}{\underline{g}(X)} = \underline{q}(X) \text{ and zero remainder}$$

$$
\begin{array}{r}
X^4+X^3+X^2 \\
X^{10}+X^8+X^5+X^2+X+1 \;\big)\; X^{14}+X^{13}+X^{11}+X^{10}+X^9+X^8+X^7+X^6+X^4+X^2 \\
X^{14}+X^{12}+X^9+X^6+X^5+X^4 \\
\hline
X^{13}+X^{12}+X^{11}+X^{10}+X^8+X^7+X^5+X^2 \\
X^{13}+X^{11}+X^8+X^5+X^4+X^3 \\
\hline
X^{12}+X^{10}+X^7+X^4+X^3+X^2 \\
X^{12}+X^{10}+X^7+X^4+X^3+X^2 \\
\hline
0 \quad \checkmark
\end{array}
$$

(b)

$$
\begin{array}{rl}
\underline{U}(x) = & X^2+X^4+X^6+X^7+X^8+X^9+X^{10}+X^{11}+X^{13}+X^{14} \\
\underline{e}(X) = & X^8+X^{10}+X^{13} \\
\hline
\underline{Z}(X) = & X^2+X^4+X^6+X^7+X^9+X^{11}+X^{14}
\end{array}
$$

(c) $\underline{Z}(x) = \underline{q}(x)\,\underline{g}(X) + \underline{S}(X)$

$$\frac{\underline{Z}(x)}{\underline{g}(X)} = \underline{q}(x) + \frac{\underline{S}(x)}{\underline{g}(x)} \leftarrow \text{remainder}$$

After performing this polynomial division, modulo-$\underline{g}(X)$, the remainder term is:

$$\underline{S}(X) = 1 + X^4 + X^6 + X^8 + X^9$$

the syndrome.

6.34 (d) $$\frac{\underline{e}(X)}{\underline{g}(X)} = [\underline{m}(X) + \underline{q}(X)] + \frac{\underline{S}(X)}{\underline{g}(X)}$$

Thus, $\underline{S}(X)$ is the remainder term when dividing $\underline{e}(X)$ by $\underline{g}(X)$. Thus when we divide $X^8 + X^{10} + X^{13}$ by $g(X)$ we get the remainder as $1 + X^4 + X^6 + X^8 + X^9$ which is the same syndrome that was computed in part (c).

(e) $$\underline{U}(X) = \underline{m}(X)\,\underline{g}(X)$$
$$\underline{Z}(X) = \underline{U}(X) + \underline{e}(X)$$
$$\underline{Z}(X) = \underline{q}(X)\,\underline{g}(X) + \underline{S}(X)$$

$$\underline{e}(X) = \underline{U}(X) + \underline{Z}(X)$$
$$= \underline{m}(X)\,\underline{g}(X) + \underline{q}(X)\,\underline{g}(X) + \underline{S}(X)$$
$$= [\underline{m}(X) + \underline{q}(X)]\,\underline{g}(X) + \underline{S}(X)$$

Syndrome $\underline{S}(X)$ obtained as the remainder of $\underline{Z}(X)$ modulo-$g(X)$ is exactly the same polynomial obtained as the remainder of $\underline{e}(X)$ modulo-$\underline{g}(X)$.

6.34 (f) Standard array dimensions for the (15,5) code:

2^5 columns

Thus of the 1024 rows — 2^{10} rows — [2^{15} entries]

We calculate, the number needed for single, double, etc., errors.

Single	Double	Triple	Quadruple
$\binom{15}{1} = 15$	$\binom{15}{2} = 105$	$\binom{15}{3} = 455$	$\binom{15}{4} = 1365$

Thus, the allocation of coset leaders to error types are:

Thus, the code is NOT a perfect code.

zero errors:	1
one error:	15
two errors:	105
three errors:	455
four errors:	448
	1024

It corrects ≈ 33% of the quadruple errors.

(g) $d_{min} \geq 2\alpha + \gamma + 1$ where α = error correction, γ = erasure correction

For the (15,5) triple-error correcting code,

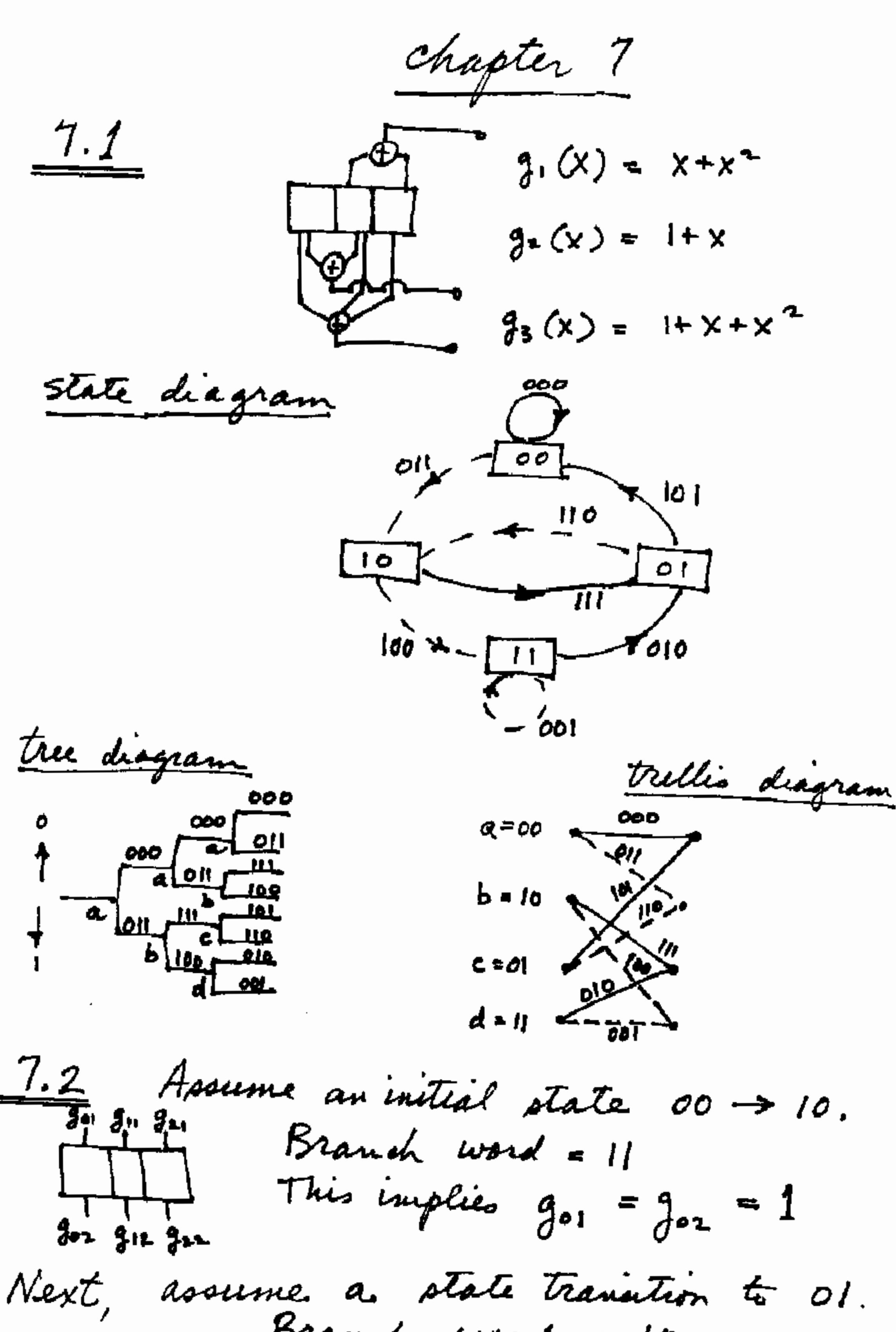
$d_{min} = 2t + 1 = 7.$

$$7 \geq 2\alpha + \gamma + 1 = 2\alpha + 3$$

Therefore $\alpha = 2$, and the code must be implemented to sacrifice error correction of 1, so that it becomes a double-error correcting and double-erasure correcting code.

Chapter 7

7.1



$g_1(X) = X + X^2$

$g_2(X) = 1 + X$

$g_3(X) = 1 + X + X^2$

7.2

Assume an initial state $00 \rightarrow 10$.

g_{01} g_{11} g_{21} / g_{02} g_{12} g_{22}

Branch word = 11

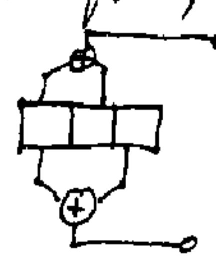
This implies $g_{01} = g_{02} = 1$

Next, assume a state transition to 01.

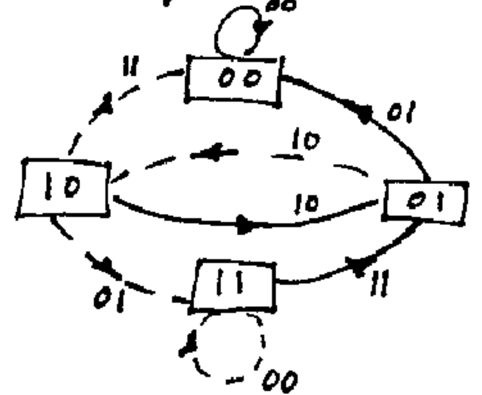
Branch word = 10

This implies $g_{11} = 1$ and $g_{12} = 0$

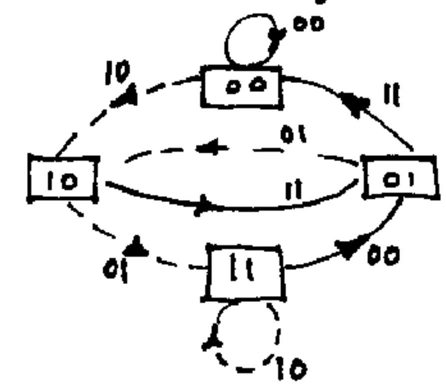
Next, assume the state transition 11 → 11. Branch word = 00. This implies $g_{21} = 0$ & $g_{22} = 1$. Therefore, the encoder connections are:



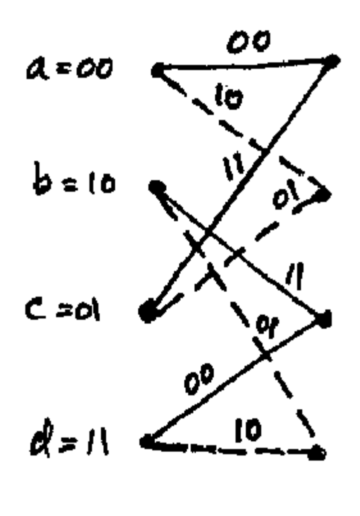
and, the completed state diagram is:



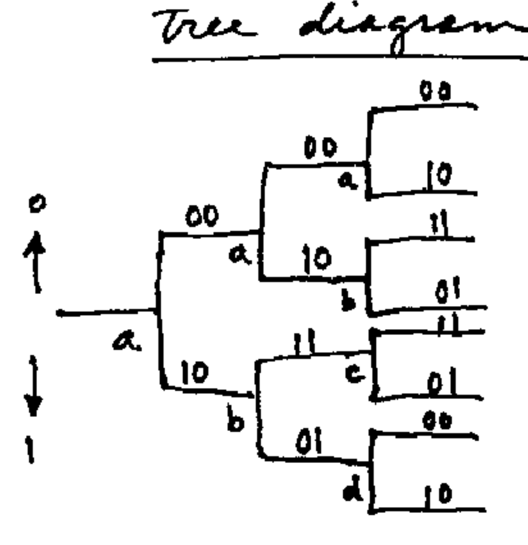
7.3 state diagram



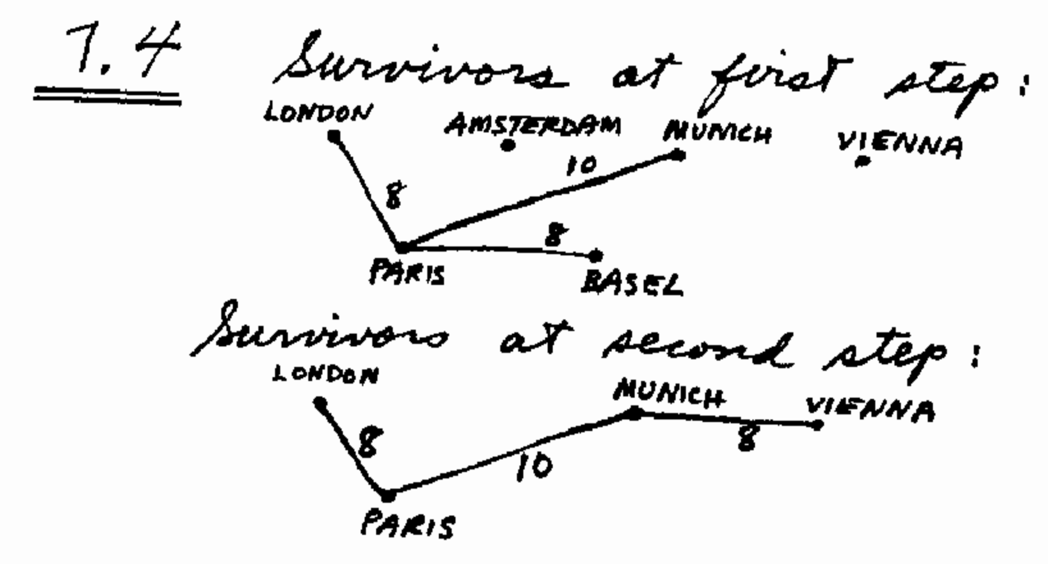
trellis diagram



Tree diagram



7.4 Survivors at first step:

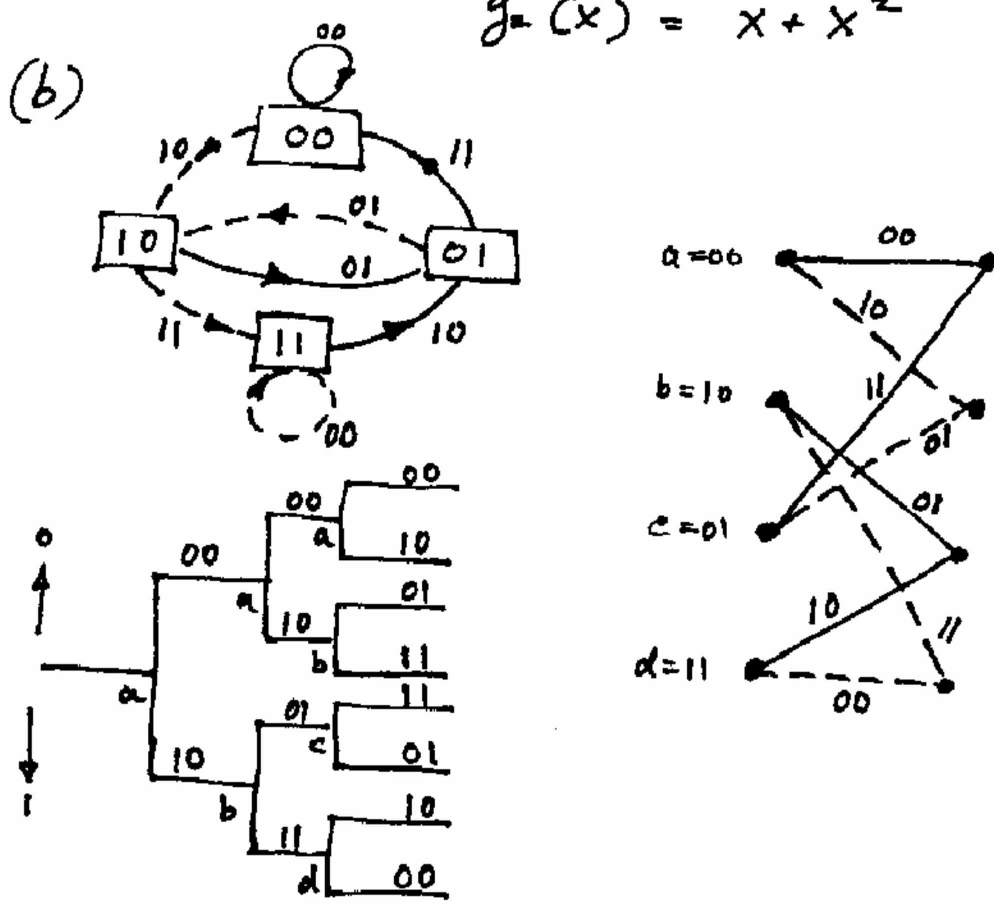


7.5 (a) vector representation: $g_1 = 101$
$g_2 = 011$

polynomial representation:

$$g_1(X) = 1 + X^2$$
$$g_2(X) = X + X^2$$

(b)



7.6

	REGISTER CONTENTS	BRANCH WORD
	100	10
	010	01
	001	11

IMPULSE RESPONSE IS: 10 01 11

INPUT (m)	OUTPUT
1	10 01 11
0	00 00 00
1	10 01 11
modulo-2 sum	10 01 01 01 11

$m(X) = 1 + X^2$; $m(X)g_1(X) =$

$$m(X)g_1(X) = (1+X^2)(1+X^2) = 1 + 0X + 0X^2 + 0X^3 + X^4$$

$$m(X)g_2(X) = (1+X^2)(X+X^2) = 0 + X + X^2 + X^3 + X^4$$

$$\text{Output } \mathbf{U}(X) = (1,0) + (0,1)X + (0,1)X^2 + (0,1)X^3 + (1,1)X^4$$

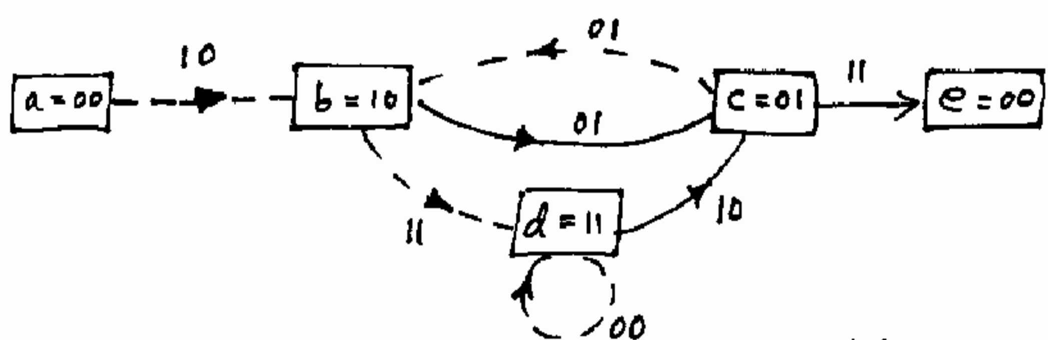
7.7 The code is catastrophic. This can be seen from the polynomial representation:

$$g_1(X) = 1 + X^2 = (1+X)(1+X)$$

$$g_2(X) = X + X^2 = X(1+X)$$

where the presence of the common factor $(1+X)$ satisfies the condition for catastrophic error propagation.

In terms of the state diagram, below



assuming the all-zeros path is the correct path, three channel errors can result in an incorrect path a, b, d, d, d, ..., d, c, e. Thus, a finite number of channel errors can cause an infinite number of decoded data bit errors.

7.8

a = 00 —10 / D→ b = 10; b = 10 —01 / D→ d = 11; b = 10 —11 / D^2→ c = 01; c = 01 —01 / D→ b = 10; d = 11 —10 / D→ d = 11; d = 11 —00 / $D^0 = 1$→ c = 01; c = 01 —11 / D^2→ e = 00

$$X_b = DX_a + DX_c$$

$$X_c = D^2X_b + X_d$$

$$X_d = DX_b + DX_d$$

$$X_e = D^2X_c$$

$$\frac{X_e}{X_a} = \frac{D^4 + D^5 - D^6}{1 - (D + D^2 + D^3 - D^4)}$$

$$= D^4 + 2D^5 + 2D^6 + \cdots$$

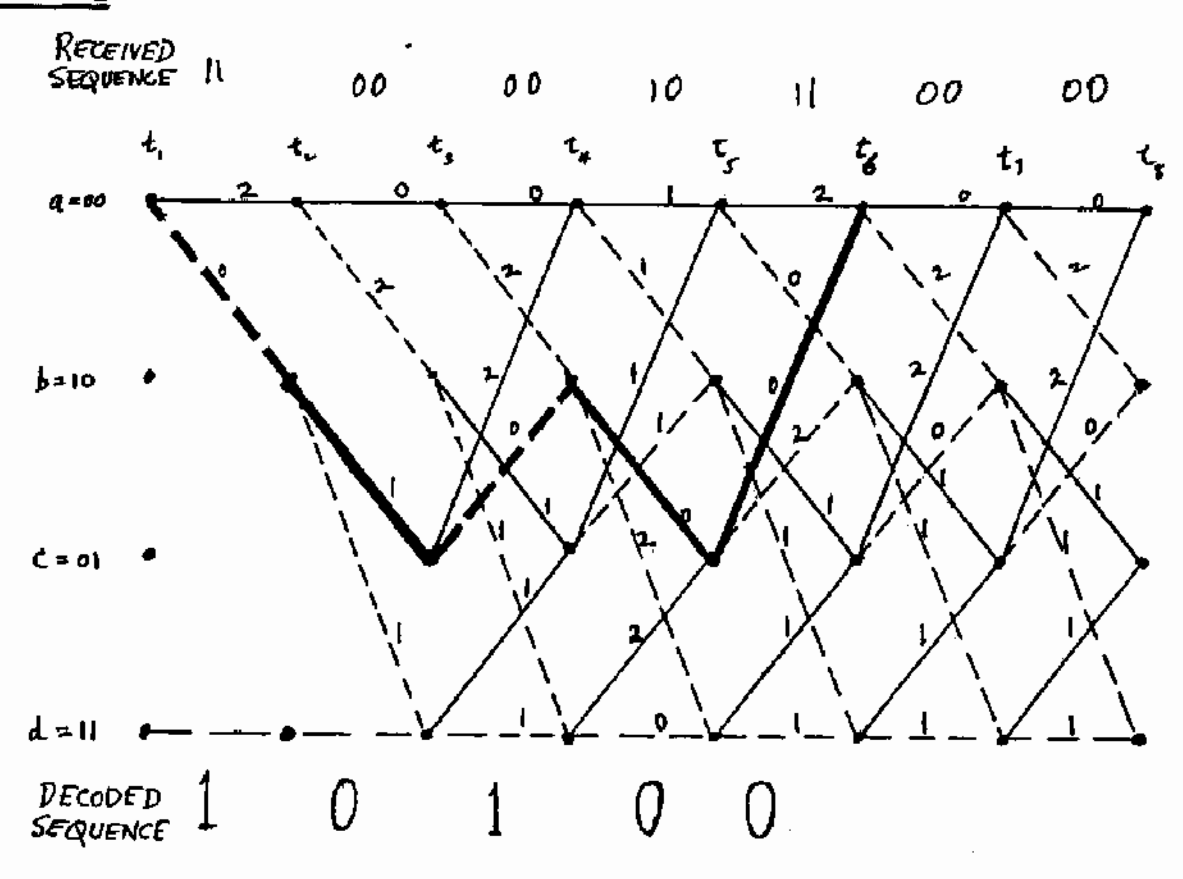
Thus, $d_f = 4$.

7.9 Hamming distance of received sequence to each of the codewords are:

distance to a = 4
" " b = 1
" " c = 5
" " d = 2

Since maximum likelihood corresponds to minimum Hamming distance for a BSC, the received sequence is decoded as codeword b.

7.10 (a)



(b) message $\underline{m}$ = 10100 would have been encoded as $\underline{U}$ = 11 10 00 10 11. Instead, the received sequence was $\underline{Z}$ = 11 00 00 10 11

↑ This bit was received in error

7.11 (a) O.K., no common factors
(b) catastrophic, factor: $(1+X)$
(c) catastrophic, factor: $(1+X^2)$
(d) O.K., no common factors
(e) catastrophic, factor: $(1+X^3)$
(f) O.K., no common factors.

7.12 (a) From Equation (6.19)

$$\left.\frac{dT(D,N)}{dN}\right|_{N=1} = \frac{D^5}{(1-2D)^2}$$

$E_b/N_0 = 6$ dB, code rate $= 1/2$, $E_c/N_0 = 3$ dB $= 2$.

$$P_B \le Q\left(\sqrt{2d_f E_c/N_0}\right) \exp\left(d_f \frac{E_c}{N_0}\right) \left.\frac{dT(D,N)}{dN}\right|_{N=1,\, D=\exp(-\frac{E_c}{N_0})}$$

from Equation (6.21). From Section 6.4.1, $d_f = 5$.

Therefore,

$$P_B \le Q\left(\sqrt{2\times5\times2}\right)\exp(5\times2)\,\frac{(e^{-2})^5}{(1-2e^{-2})^2}$$

$$= Q\left(\sqrt{20}\right) e^{10} \left(8.535\times10^{-5}\right)$$

$$= Q(4.47) \times 1.88$$

$$\cong \frac{1}{4.47\sqrt{2\pi}} e^{-10} \times 1.88 = 7.6\times10^{-6}$$

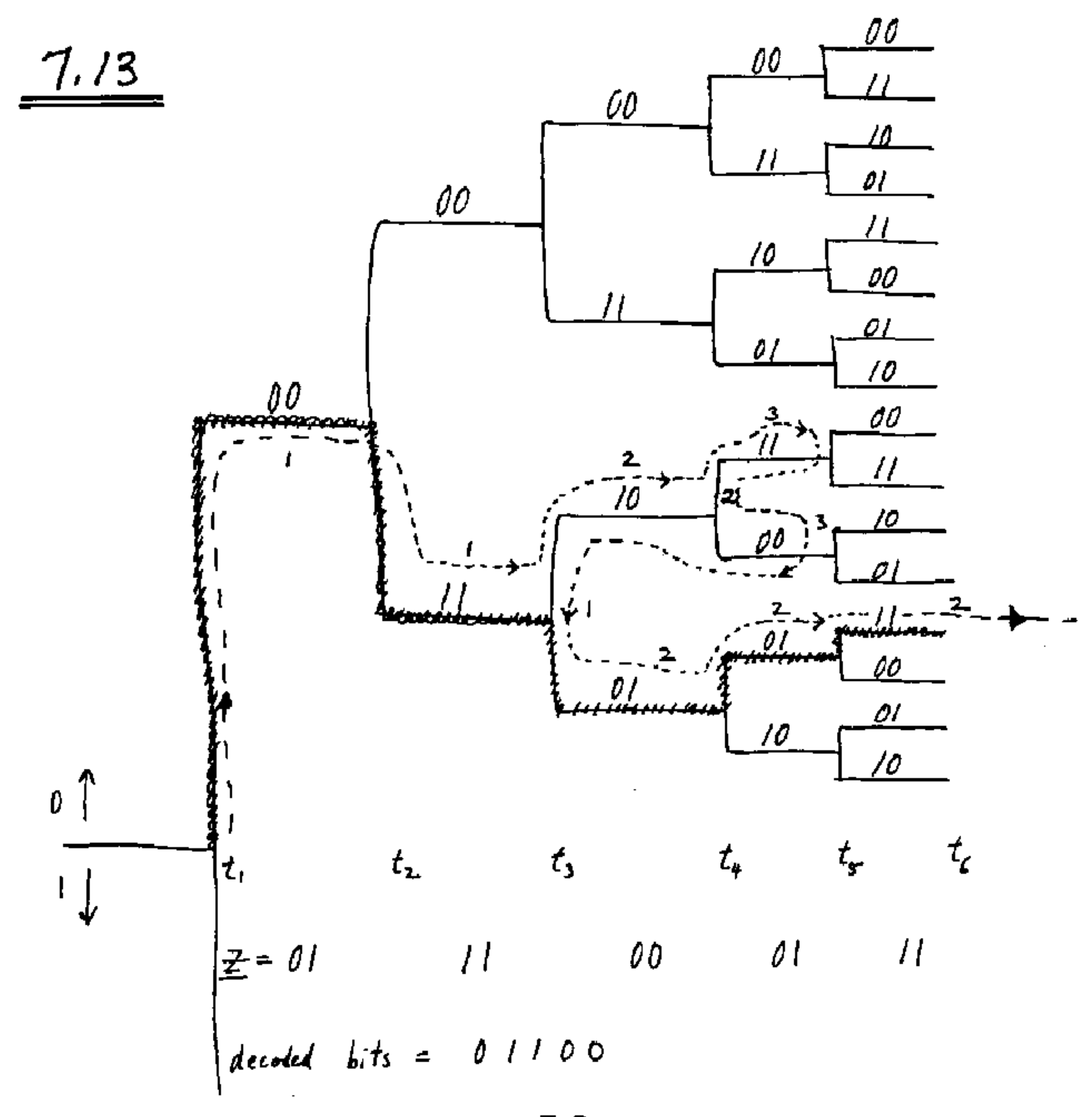
(b) uncoded case:

$$P_B = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) = Q\left(\sqrt{2\times 3.98}\right) = Q(2.82)$$

Using Table 3.1, $Q(2.82) = 2.4\times 10^{-3}$

$$\left.\begin{array}{l}\text{PERFORMANCE}\\ \text{IMPROVEMENT}\end{array}\right\} = \frac{2.4\times 10^{-3}}{7.6\times 10^{-6}} = 315.8$$

7.13



$\underline{Z}$ = 01 11 00 01 11

decoded bits = 0 1 1 0 0

7.14 Received sequence $\underline{Z}$ = 01 11 00 01 11
The paths of the first 3 branches of the tree are compared with the first 6 bits of $\underline{Z}$

upper-half metrics: 3, 5, 2, 2
lower-half metrics: 4, 2, 3, 3

There is a tie for the minimum metric. The upper half is arbitrarily chosen. Thus, the first decoded bit is "zero". Continuing one branch deeper into the tree

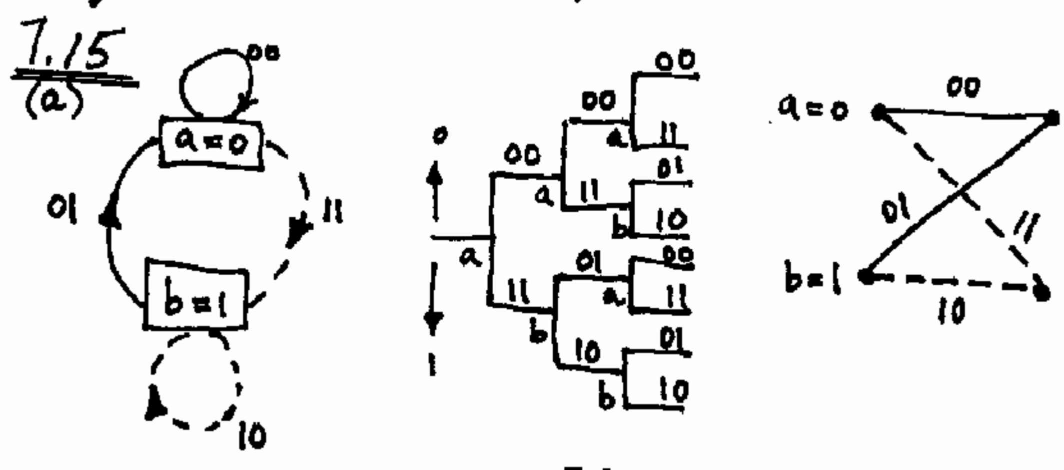
upper-half metrics: 3, 3, 6, 4
lower-half metrics: 2, 2, 1, 3

Hence the second decoded bit is a "one". continuing, yields

upper-half metrics: 4, 2, 3, 3
lower-half metrics: 1, 3, 4, 4

Thus, the third decoded bit is a "one". Adding zeros to $\underline{Z}$ to decode the fourth and fifth bits finally yields the decoded sequence = 01100.

7.15
(a)



(b) Received sequence $\mathbf{Z}$ = 110010

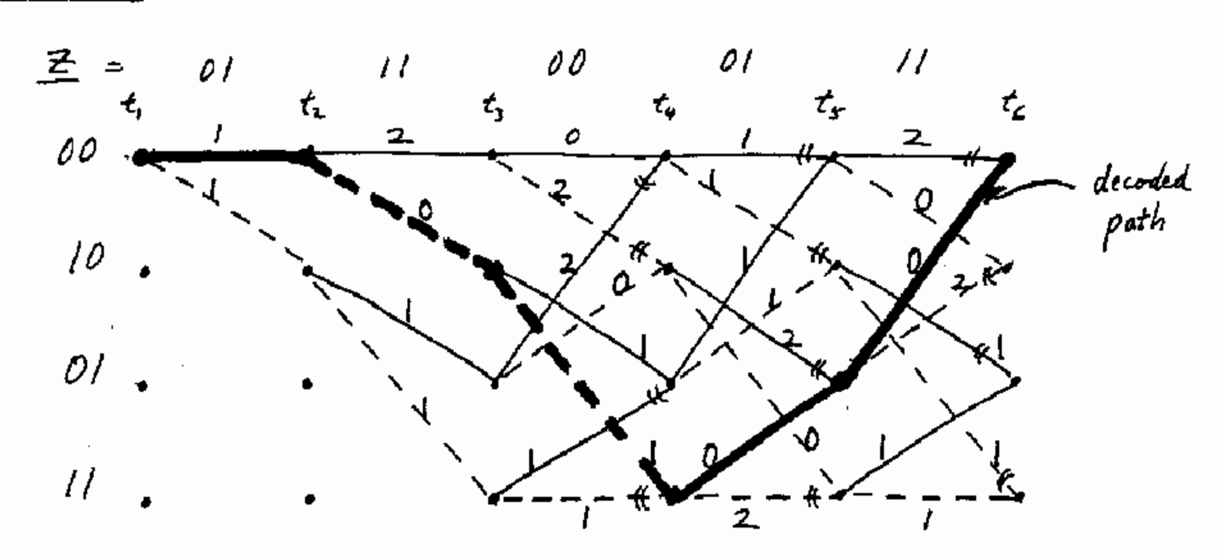
upper-half metrics: 2, 4
lower-half metrics: 1, 1

First decoded bit is "1". Continuing,

upper-half metrics: 2, 2
lower-half metrics: 3, 1

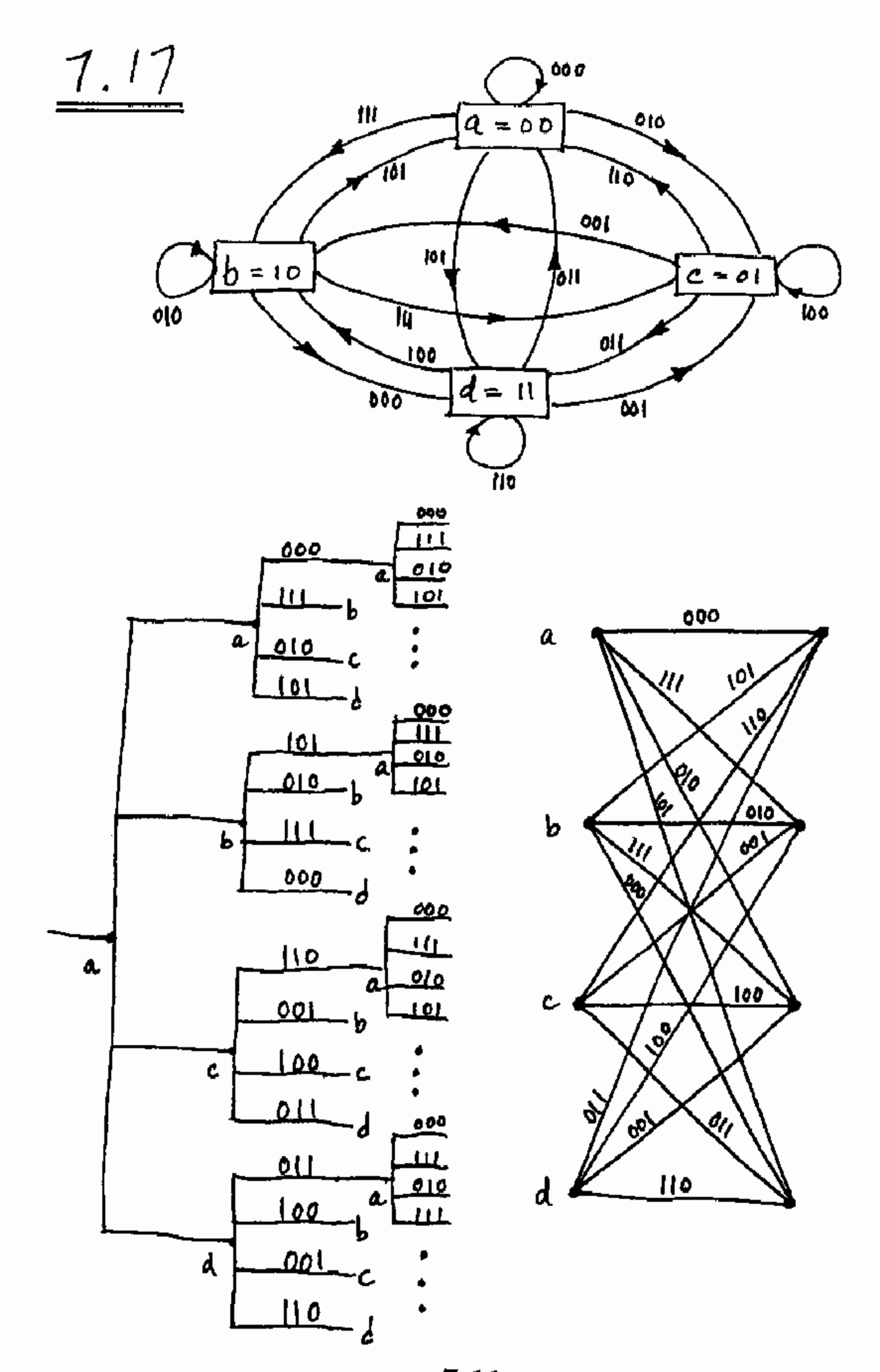
Second decoded bit is "1". Adding zeros to $\mathbf{Z}$ to decode the third bit, yields the decoded sequence = 111.

7.16



The decoded sequence is: 01100 which agrees with the decoded sequence from the sequential decoder of Problem 7.13 and the feedback decoder of Problem 7.14.

7.17



7.18

$$P_b = 2000\, P_B^4 \;;\quad P_B = \left(\frac{P_b}{2000}\right)^{1/4} = 8.4 \times 10^{-3}$$

$$P_B = 8.4 \times 10^{-3} = \frac{1}{2} \exp\left(-\frac{1}{2} E_b/N_0\right)$$

$$E_b/N_0 = -2 \ln\left(2 P_B\right) = 8.17 = 9.12 \text{ dB}$$

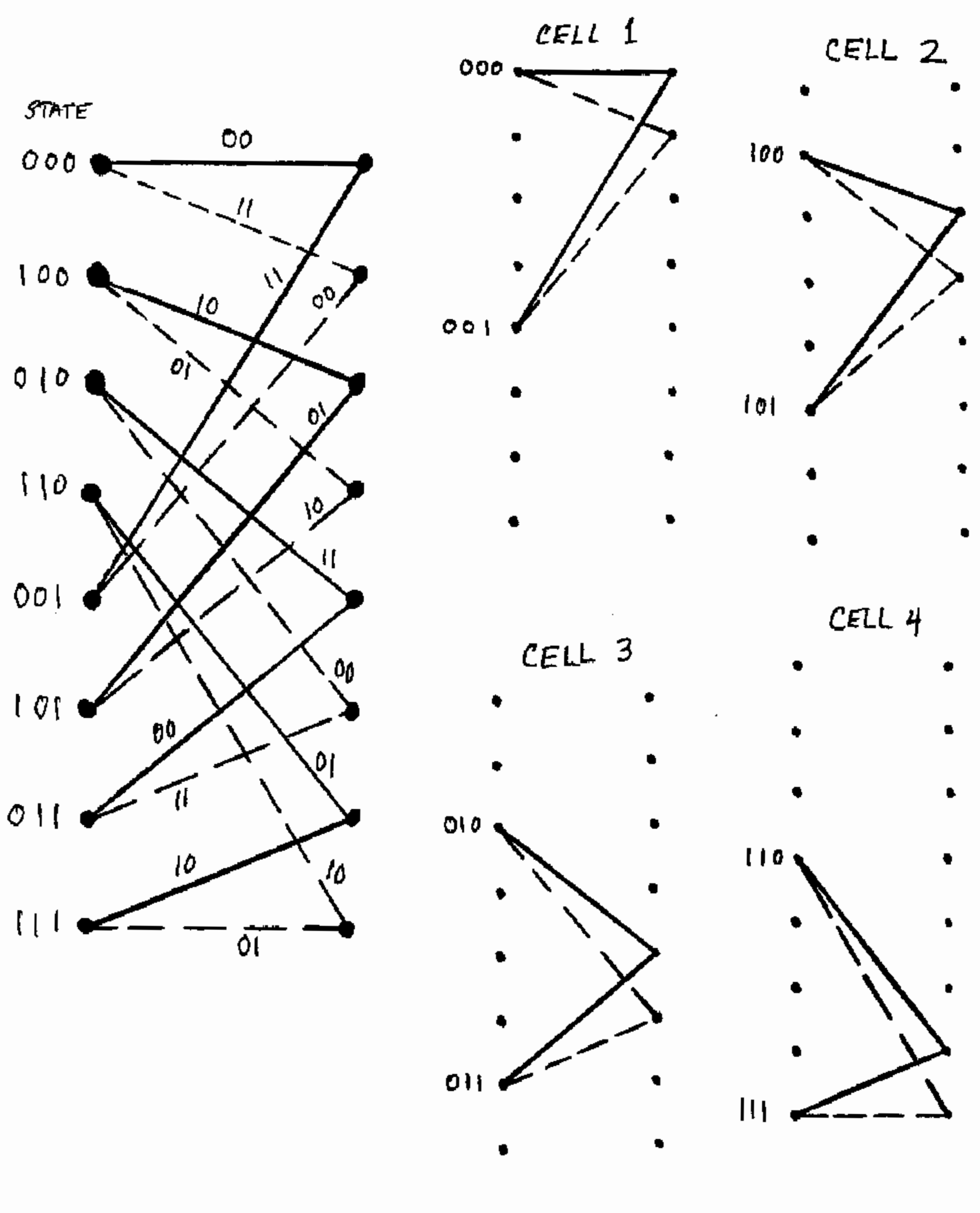
$$\frac{P_r}{N_0}(\text{dB}) = \frac{E_b}{N_0}(\text{dB}) + R(\text{dB-bit/s}) = 9.12 + 60 = 69.1 \text{ dB}$$

7.19 (a)

$K = 4$
rate $= 1/2$
encoder
from Table 7.4

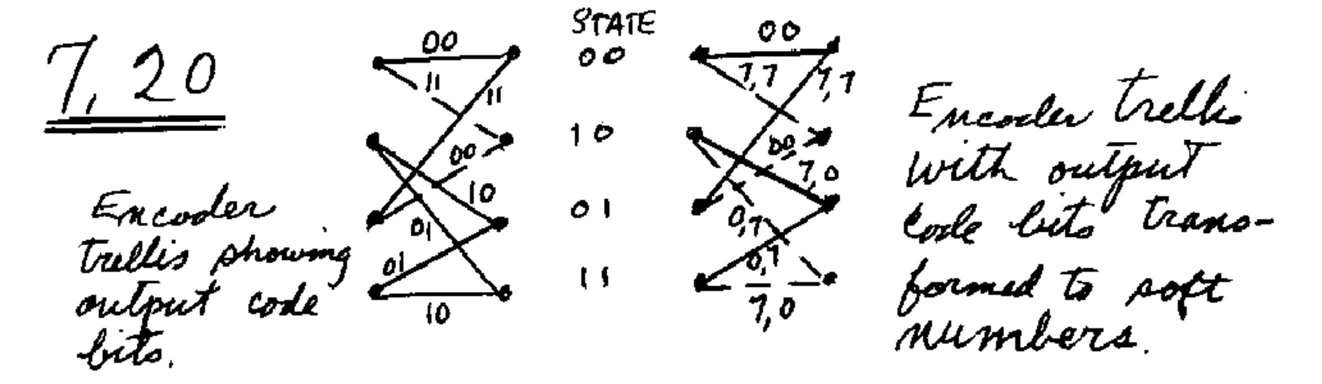
	U_1 U_2
1 0 0 0	1 1
1 1 0 0	0 1
1 1 1 0	1 0
0 1 1 1	1 0
1 0 1 1	1 1
0 1 0 1	0 1
0 0 1 0	1 1
0 0 0 1	1 1
0 0 0 0	0 0

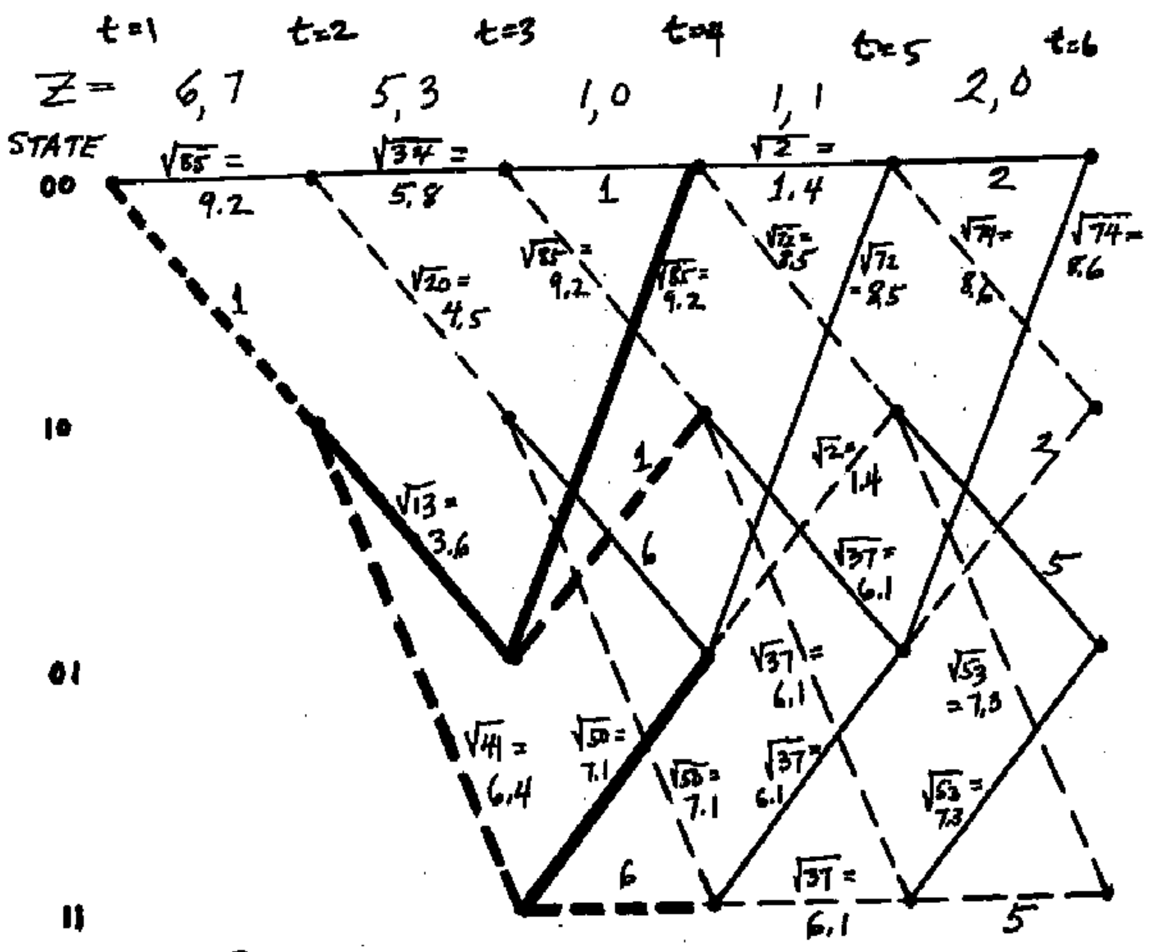
7.19 (b) and (c)



TRELLIS
CELLS
STATE
000
100
010
110
001
101
011
111
00
11
11
00
10
01
01
10
11
00
00
01
11
10
10
01
CELL 1
000
001
CELL 2
100
101
CELL 3
010
011
CELL 4
110
111

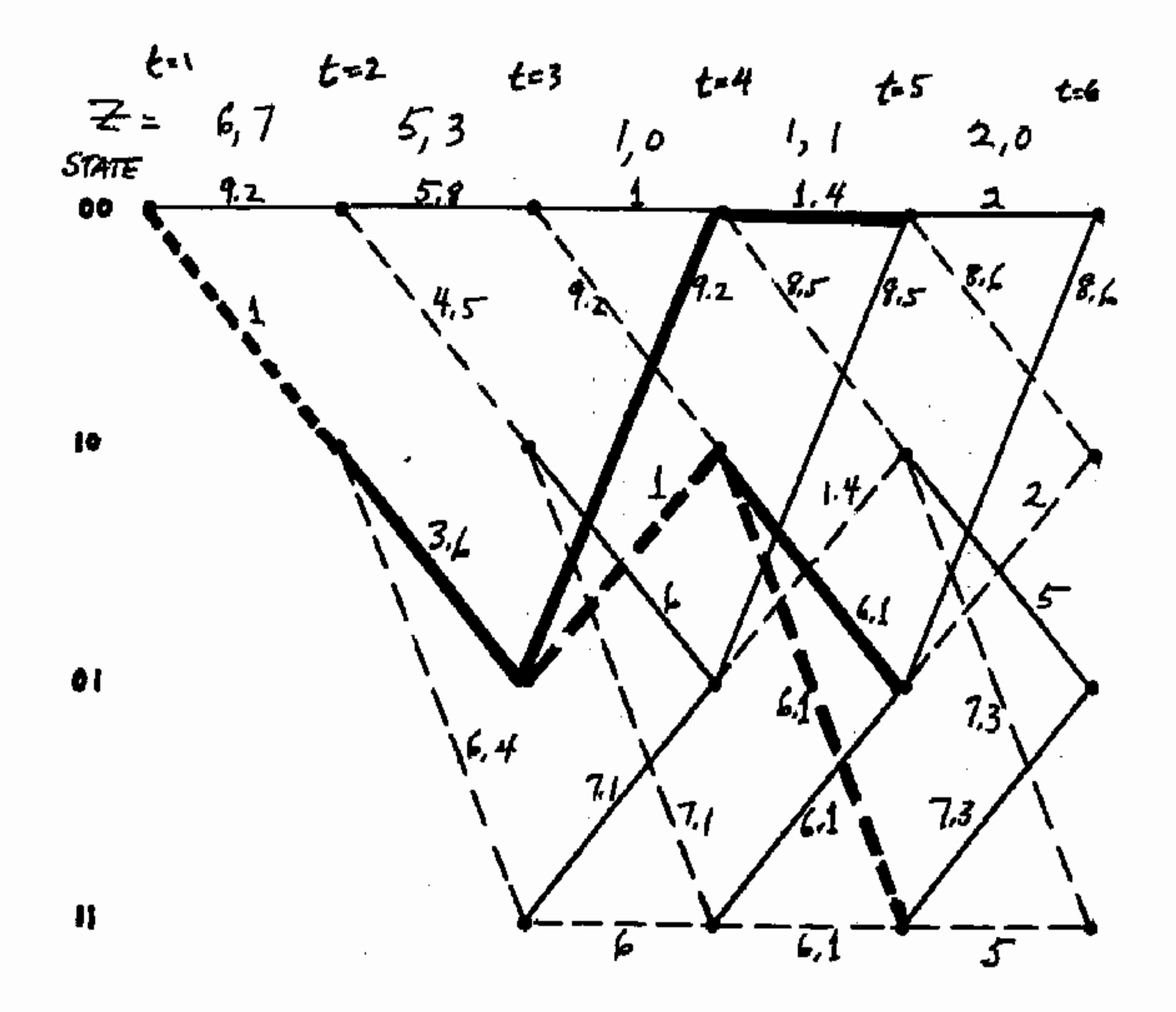
7.20





Decoder Trellis — showing Euclidean distance metrics. Remergence at time $t = 4$, allows use of the Viterbi algorithm, resulting in a common stem between $t = 1$ and $t = 2$. The first bit is decoded as a "1".

7.20 (cont'd.)



Remergence at time $t=5$ results in a common stem between $t=2$ and $t=3$. The second bit is decoded as a "0". Hence, the first two decoded bits are: 1, 0.

8.1

The polynomials in parts (a), (d), (g), and (h) are primitive. The rest are not primitive. We show the solution to part (a) in the classical way – that is, an irreducible polynomial, $f(X)$, of degree m is said to be primitive, if the smallest positive integer n for which $f(X)$ divides $X^n + 1$ is $n = 2^m - 1$. Thus, for part (a), we verify that this degree $m = 3$ polynomial is primitive by determining that it divides $X^n + 1 = X^{(2^m-1)} + 1 = X^7 + 1$, but does not divide $X^n + 1$, for values of n in the range of $1 \leq n < 7$. Below, we show that $X^3 + X^2 + 1$ divides $X^7 + 1$.

$$\begin{array}{r}
X^4 + X^3 + X^2 + 1 \\
X^3 + X^2 + 1 \overline{\big) \; X^7 + 1} \\
\underline{X^7 + X^6 + X^4} \\
X^6 + X^4 + 1 \\
\underline{X^6 + X^5 + X^3} \\
X^5 + X^4 + X^3 + 1 \\
\underline{X^5 + X^4 + X^2} \\
X^3 + X^2 + 1 \\
\underline{X^3 + X^2 + 1} \\
0
\end{array}$$

Next we exhaustively check to see that the remaining conditions are met.

$$\begin{array}{r}
X^3 + X^2 + X \\
X^3 + X^2 + 1 \overline{\big) X^6 + 1} \\
\underline{X^6 + X^5 + X^3} \\
X^5 + X^3 + 1 \\
\underline{X^5 + X^4 + X^2} \\
X^4 + X^3 + X^2 + 1 \\
\underline{X^4 + X^3 + X} \\
X^2 + X + 1
\end{array}$$

$$
\begin{array}{r}
X^2+X+1 \\
X^3+X^2+1 \overline{)X^5+1} \\
\underline{X^5+X^4+X^2} \\
X^4+X^2+1 \\
\underline{X^4+X^3+X} \\
X^3+X^2+X+1 \\
\underline{X^3+X^2+1} \\
X
\end{array}
$$

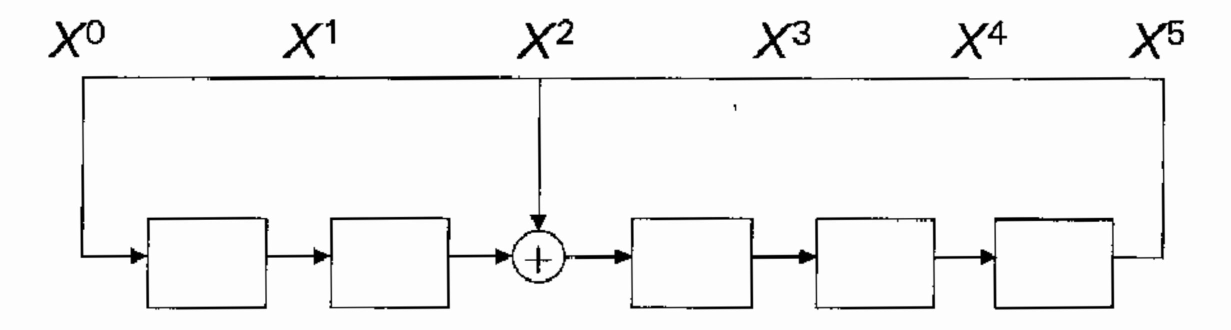
$$
\begin{array}{r}
X+1 \\
X^3+X^2+1 \overline{)X^4+1} \\
\underline{X^4+X^3+X} \\
X^3+X+1 \\
\underline{X^3+X^2+1} \\
X^2+X+1
\end{array}
$$

$$
\begin{array}{r}
1 \\
X^3+X^2+1 \overline{)X^3+1} \\
\underline{X^3+X^2+1} \\
X^2
\end{array}
$$

The remaining conditions are met, since we have shown that X^3+X^2+1 does not divide $X^n + 1$, for values of n in the range of $1 \leq n < 7$.

Next we use a LFSR to illustrate an easier way of determining whether a polynomial is primitive. As an example we use this method to verify that the part (g) polynomial is primitive.
We draw the LFSR (shown below), with the feedback connections corresponding to the coefficients of the polynomial $1 + X^2 + X^5$ similar to the example of Figure 8.8. We load into the circuit-registers any nonzero setting, say 1 0 0 0 0, and perform a right shift with each clock pulse. For this polynomial, the circuit generates each of the nonzero field elements within one period (as seen in the table below), hence the polynomial which defines this $GF(2^5)$ field is a primitive polynomial.

Lowest order stage				Highest order stage	Decimal equivalent
1	0	0	0	0	16
0	1	0	0	0	8
0	0	1	0	0	4
0	0	0	1	0	2
0	0	0	0	1	1
1	0	1	0	0	20
0	1	0	1	0	10
0	0	1	0	1	5
1	0	1	1	0	22
0	1	0	1	1	11
1	0	0	0	1	17
1	1	1	0	0	28
0	1	1	1	0	14
0	0	1	1	1	7
1	0	1	1	1	23
1	1	1	1	1	31
1	1	0	1	1	27
1	1	0	0	1	25
1	1	0	0	0	24
0	1	1	0	0	12
0	0	1	1	0	6
0	0	0	1	1	3
1	0	1	0	1	21
1	1	1	1	0	30
0	1	1	1	1	15
1	0	0	1	1	19
1	1	1	0	1	29
1	1	0	1	0	26
0	1	1	0	1	13
1	0	0	1	0	18
0	1	0	0	1	9
1	0	0	0	0	16



LFSR with feedback connections corresponding to the coefficients of the polynomial $1 + X^2 + X^5$

8.2 (a)

$$t = \frac{n-k}{2} = \frac{7-3}{2} = 2 \text{ symbols}$$

$$(n, k) = (2^m - 1, 2^m - 1 - 2t)$$

$$2^m - 1 = 7, \quad m = 3 \text{ bits/symbol}$$

(b) The (7, 3) R-S code has $2^{km} = 2^9 = 512$ codewords out of a total of $2^{nm} = 2^{21} = 2{,}097{,}152$ possible binary words. We therefore know that the dimensions of the standard array must contain 2^{21} total entries and 2^9 columns (one for each codeword). Thus, we can compute that the number of rows must be $2^{21}/2^9 = 2^{12}$ (4096 rows).

(c) and **(d)** The codeword is made up of seven symbols, each symbol containing 3 bits. How many ways are there to make a symbol error, given that a symbol is made up of 3 bits? There are $\binom{7}{1} + \binom{7}{2} + \binom{7}{3} = 7$ ways to make an error in any one of the symbols. Next, we ask how many ways are there to make single symbol errors in a seven-symbol codeword, given that we have just computed that there are seven ways to make an error in any one symbol? There are $\binom{7}{1} \times 7 = 49$ ways to

make single-symbol errors. How many ways are there to make double-symbol errors? There are $\binom{7}{2} \times 7 \times 7 = 1029$ ways to make double symbol errors? How many ways are there to make triple-symbol errors? There are $\binom{7}{3} \times 7 \times 7 \times 7 = 12{,}005$. How do we use this information together with the dimensions of the standard array to corroborate (and gain some insight into) the part (a) finding that the (7, 3) R-S code is a double symbol error correcting code? There are 4096 rows in the standard array of this example, and the all-zeros vector occupies the first row. Of the 4095 remaining rows, 49 are allocated to single symbol errors, 1029 are allocated to double-symbol errors, and thus the code is not a perfect code because there remains a residual 3015 rows that can be allocated to triple-symbol errors.

8.3

From Table 8.1, we select the primitive polynomial $1 + x + x^4$, and map the field elements versus basis elements as follows:

$1 + \alpha + \alpha^4 = 0, \quad \alpha^4 = -1 - \alpha, \quad \alpha^4 = 1 + \alpha$

$\alpha^5 = \alpha\,\alpha^4 = \alpha\,(1 + \alpha) = \alpha + \alpha^2$

$\alpha^6 = \alpha\,(\alpha + \alpha^2) = \alpha^2 + \alpha^3$

$\alpha^7 = \alpha\,(\alpha^2 + \alpha^3) = \alpha^3 + \alpha^4 = 1 + \alpha + \alpha^3$

$\alpha^8 = \alpha\,\alpha^7 = \alpha\,(1 + \alpha + \alpha^3) = \alpha + \alpha^2 + \alpha^4 = \alpha + \alpha^2 + 1 + \alpha = 1 + \alpha^2$

$\alpha^9 = \alpha\,\alpha^8 = \alpha\,(1 + \alpha^2) = \alpha + \alpha^3$

$\alpha^{10} = \alpha\,\alpha^9 = \alpha\,(\alpha + \alpha^3) = \alpha^2 + \alpha^4 = 1 + \alpha + \alpha^2$

$\alpha^{11} = \alpha\,\alpha^{10} = \alpha\,(1 + \alpha + \alpha^2) = \alpha + \alpha^2 + \alpha^3$

$\alpha^{12} = \alpha\,\alpha^{11} = \alpha\,(\alpha + \alpha^2 + \alpha^3) = \alpha + \alpha^2 + \alpha^4 = \alpha + \alpha^2 + 1 + \alpha = 1 + \alpha^2$

	X^0	X^1	X^2	X^3
0	0	0	0	0
α^{0}	1	0	0	0
α^{1}	0	1	0	0
α^{2}	0	0	1	0
α^{3}	0	0	0	1
α^{4}	1	1	0	0
α^{5}	0	1	1	0
α^{6}	0	0	1	1
α^{7}	1	1	0	1
α^{8}	1	0	1	0
α^{9}	0	1	0	1
α^{10}	1	1	1	0
α^{11}	0	1	1	1
α^{12}	1	1	1	1
α^{13}	1	0	1	1
α^{14}	1	0	0	1

Because of symmetry, we show only the triangular half of the tables.

Addition Table

+		0	α^{0}	α^{1}	α^{2}	α^{3}	α^{4}	α^{5}	α^{6}	α^{7}	α^{8}	α^{9}	α^{10}	α^{11}	α^{12}	α^{13}	α^{14}
0		0	α^{0}	α^{1}	α^{2}	α^{3}	α^{4}	α^{5}	α^{6}	α^{7}	α^{8}	α^{9}	α^{10}	α^{11}	α^{12}	α^{13}	α^{14}
α^{0}			0	α^{4}	α^{8}	α^{14}	α^{1}	α^{10}	α^{13}	α^{9}	α^{2}	α^{7}	α^{5}	α^{12}	α^{11}	α^{6}	α^{3}
α^{1}				0	α^{5}	α^{9}	α^{0}	α^{2}	α^{11}	α^{14}	α^{10}	α^{3}	α^{8}	α^{6}	α^{13}	α^{12}	α^{7}
α^{2}					0	α^{6}	α^{10}	α^{1}	α^{3}	α^{12}	α^{0}	α^{11}	α^{4}	α^{9}	α^{7}	α^{14}	α^{13}
α^{3}						0	α^{7}	α^{11}	α^{2}	α^{4}	α^{13}	α^{1}	α^{12}	α^{5}	α^{10}	α^{8}	α^{0}
α^{4}							0	α^{8}	α^{12}	α^{3}	α^{5}	α^{14}	α^{2}	α^{13}	α^{6}	α^{11}	α^{9}
α^{5}								0	α^{9}	α^{13}	α^{4}	α^{6}	α^{0}	α^{3}	α^{14}	α^{7}	α^{12}
α^{6}									0	α^{10}	α^{14}	α^{5}	α^{7}	α^{1}	α^{4}	α^{0}	α^{8}
α^{7}										0	α^{11}	α^{0}	α^{6}	α^{8}	α^{2}	α^{5}	α^{1}
α^{8}											0	α^{12}	α^{1}	α^{7}	α^{9}	α^{3}	α^{6}
α^{9}												0	α^{13}	α^{2}	α^{8}	α^{10}	α^{4}
α^{10}													0	α^{14}	α^{3}	α^{9}	α^{11}
α^{11}														0	α^{0}	α^{4}	α^{10}
α^{12}															0	α^{1}	α^{5}
α^{13}																0	α^{2}
α^{14}																	0

Multiplication table

×		0	α^0	α^1	α^2	α^3	α^4	α^5	α^6	α^7	α^8	α^9	α^{10}	α^{11}	α^{12}	α^{13}	α^{14}
0		0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
α^0			α^0	α^1	α^2	α^3	α^4	α^5	α^6	α^7	α^8	α^9	α^{10}	α^{11}	α^{12}	α^{13}	α^{14}
α^1				α^2	α^3	α^4	α^5	α^6	α^7	α^8	α^9	α^{10}	α^{11}	α^{12}	α^{13}	α^{14}	α^0
α^2					α^4	α^5	α^6	α^7	α^8	α^9	α^{10}	α^{11}	α^{12}	α^{13}	α^{14}	α^0	α^1
α^3						α^6	α^7	α^8	α^9	α^{10}	α^{11}	α^{12}	α^{13}	α^{14}	α^0	α^1	α^2
α^4							α^8	α^9	α^{10}	α^{11}	α^{12}	α^{13}	α^{14}	α^0	α^1	α^2	α^3
α^5								α^{10}	α^{11}	α^{12}	α^{13}	α^{14}	α^0	α^1	α^2	α^3	α^4
α^6									α^{12}	α^{13}	α^{14}	α^0	α^1	α^2	α^3	α^4	α^5
α^7										α^{14}	α^0	α^1	α^2	α^3	α^4	α^5	α^6
α^8											α^1	α^2	α^3	α^4	α^5	α^6	α^7
α^9												α^3	α^4	α^5	α^6	α^7	α^8
α^{10}													α^5	α^6	α^7	α^8	α^9
α^{11}														α^7	α^8	α^9	α^{10}
α^{12}															α^9	α^{10}	α^{11}
α^{13}																α^{11}	α^{12}
α^{14}																	α^{13}

8.4

$$\alpha^5X^2+X+\alpha^4$$

$$X^4+\alpha^3X^3+\alpha^0X^2+\alpha^1X+\alpha^3 \overline{\big)\alpha^5X^6+\alpha^3X^5+\alpha^1X^4}$$

$$\underline{\alpha^5X^6+\alpha X^5+\alpha^5X^4+\alpha^6X^3+\alpha X^2}$$

$$\left(\alpha^3+\alpha\right)X^5+\left(\alpha^5+\alpha\right)X^4+\alpha^6X^3+\alpha X^2$$

$$\alpha^0X^5+\alpha^6X^4+\alpha^6X^3+\alpha X^2$$

$$\underline{X^5+\alpha^3X^4+\alpha^0X^3+\alpha X^2+\alpha^3X}$$

$$\left(\alpha^6+\alpha^3\right)X^4+\left(\alpha^0+\alpha^6\right)X^3+\alpha^3X$$

$$\alpha^4X^4+\alpha^2X^3+\alpha^3X$$

$$\underline{\alpha^4X^4+\alpha^4\alpha^3X^3+\alpha^4X^2+\alpha^1\alpha^4X+\alpha^3\alpha^4}$$

$$\left(\alpha^2+1\right)X^3+\alpha^4X^2+\left(\alpha^3+\alpha^5\right)X+1$$

$$\alpha^6X^3+\alpha^4X^2+\alpha^2X+1$$

8.4 (cont'd.)

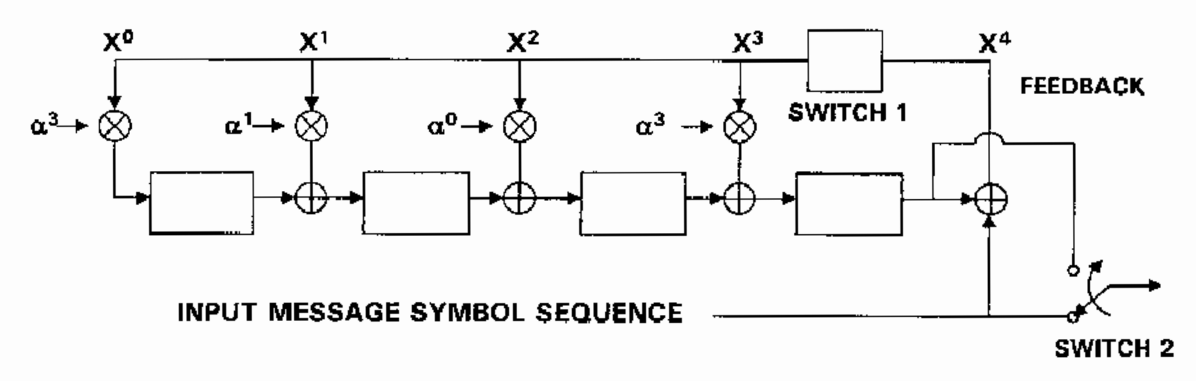
$$\text{parity}\quad \mathbf{p}(X) = X^{n-k} m(X) \ \text{modulo}\, \mathbf{g}(X)$$

$$\text{remainder} = \text{parity} = 1 + \alpha^2 X + \alpha^4 X^2 + \alpha^6 X^3$$

$$\mathbf{U}(X) = 1 + \alpha^2 X + \alpha^4 X^2 + \alpha^6 X^3 + \alpha^1 X^4 + \alpha^3 X^5 + \alpha^5 X^6$$

= 100 001 011 101 (parity) 010 110 111 (data)

8.5 (a) For the (7, 3) R-S code, we use the LFSR from Figure 8.9.



With Figure 8.7, we transform the message symbols {6, 5, 1} to $\alpha^3\,\alpha^6\,\alpha^2$, where the rightmost symbol is the earliest.

Input	Clock	Register				Feedback
$\alpha^3\alpha^6\alpha^2$	0	0	0	0	0	α^2
$\alpha^3\alpha^6$	1	$\frac{\alpha^2\alpha^3}{\alpha^5}$	$\frac{\alpha^2\alpha^1}{\alpha^3}$	$\frac{\alpha^2\alpha^0}{\alpha^2}$	$\frac{\alpha^2\alpha^3}{\alpha^5}$	$\frac{\alpha^5\alpha^6}{\alpha^1}$
α^3	2	$\frac{\alpha^1\alpha^3}{\alpha^4}$	$\frac{\alpha^1\alpha^1+\alpha^5}{\alpha^3}$	$\frac{\alpha^1\alpha^0+\alpha^3}{\alpha^0}$	$\frac{\alpha^1\alpha^3+\alpha^2}{\alpha^1}$	$\frac{\alpha^1+\alpha^3}{\alpha^0}$
	3	$\frac{\alpha^0\alpha^3}{\alpha^3}$	$\frac{\alpha^0\alpha^1+\alpha^4}{\alpha^2}$	$\frac{\alpha^0\alpha^0+\alpha^3}{\alpha^1}$	$\frac{\alpha^0\alpha^3+\alpha^0}{\alpha^1}$	–

$\alpha^3\ \alpha^2\ \alpha^1\ \alpha^1$ (parity) $\alpha^3\ \alpha^6\ \alpha^2$ (data)

Codeword = 110 001 010 010 110 101 001

8.5 (b)

$$\mathbf{U}(X) = \alpha^3 + \alpha^2 X + \alpha^1 X^2 + \alpha^1 X^3 + \alpha^3 X^4 + \alpha^6 X^5 + \alpha^2 X^6$$
$$\mathbf{U}(\alpha) = \alpha^3 + \alpha^3 + \alpha^3 + \alpha^4 + \alpha^0 + \alpha^4 + \alpha^1 + \alpha^1 + \alpha^1 = 0$$
$$\mathbf{U}(\alpha^2) = \alpha^3 + \alpha^4 + \alpha^5 + \alpha^0 + \alpha^4 + \alpha^2 + \alpha^0 + \alpha^2 + \alpha^2 = 0$$
$$\mathbf{U}(\alpha^3) = \alpha^3 + \alpha^5 + \alpha^0 + \alpha^3 + \alpha^1 + \alpha^0 + \alpha^6 + \alpha^6 + \alpha^6 = 0$$
$$\mathbf{U}(\alpha^4) = \alpha^3 + \alpha^6 + \alpha^2 + \alpha^6 + \alpha^5 + \alpha^5 + \alpha^5 + \alpha^5 + \alpha^5 = 0$$

Hence, $\mathbf{U}(X)$ is a valid codeword because the syndrome yields an all-zeros result when evaluated at the roots of the generator polynomial.

8.6 (a)

For this example, the error pattern can be described as

$$\mathbf{e}(X) = \sum_{n=0}^{6} e_n X^n$$
$$= (000) + (000)X + (000)X^2 + (000)X^3 + (000)X^4 + (111)X^5 + (111)X^6$$

Using $\mathbf{U}(X)$ from Problem 8.5, the received polynomial can be written:

$\mathbf{r}(X) = \mathbf{U}(X) + \mathbf{e}(X)$

$$= \alpha^3 + \alpha^2 X + \alpha^1 X^2 + \alpha^1 X^3 + \alpha^3 X^4 + \alpha^6 X^5 + \alpha^2 X^6 + \alpha^5 X^5 + \alpha^5 X^6$$
$$= \alpha^3 + \alpha^2 X + \alpha^1 X^2 + \alpha^1 X^3 + \alpha^3 X^4 + \alpha^1 X^5 + \alpha^3 X^6$$

We find the syndrome values by evaluating $\mathbf{r}(X)$ at the roots of $\mathbf{g}(X)$:

$$\mathbf{r}(\alpha) = \alpha^3 + \alpha^3 + \alpha^3 + \alpha^4 + \alpha^0 + \alpha^6 + \alpha^2 = \alpha^6$$
$$\mathbf{r}(\alpha^2) = \alpha^3 + \alpha^4 + \alpha^5 + \alpha^0 + \alpha^4 + \alpha^4 + \alpha^1 = \alpha^0$$
$$\mathbf{r}(\alpha^3) = \alpha^3 + \alpha^5 + \alpha^0 + \alpha^3 + \alpha^1 + \alpha^2 + \alpha^0 = \alpha^0$$
$$\mathbf{r}(\alpha^4) = \alpha^3 + \alpha^6 + \alpha^2 + \alpha^6 + \alpha^5 + \alpha^0 + \alpha^6 = \alpha^2$$

(b)

$$\mathbf{e}(X) = \alpha^5 X^5 + \alpha^5 X^6$$
$$\mathbf{e}(\alpha) = \alpha^3 + \alpha^4 = \alpha^6$$
$$\mathbf{e}(\alpha^2) = \alpha^1 + \alpha^3 = \alpha^0$$
$$\mathbf{e}(\alpha^3) = \alpha^6 + \alpha^2 = \alpha^0$$
$$\mathbf{e}(\alpha^4) = \alpha^4 + \alpha^1 = \alpha^2$$

8.7

Using the autoregressive model of Equation (8.40)

$$\begin{bmatrix} S_1 & S_2 \\ S_2 & S_3 \end{bmatrix} \begin{bmatrix} \sigma_2 \\ \sigma_1 \end{bmatrix} = \begin{bmatrix} S_3 \\ S_4 \end{bmatrix} \qquad \begin{bmatrix} \alpha^6 & \alpha^0 \\ \alpha^0 & \alpha^0 \end{bmatrix} \begin{bmatrix} \sigma_2 \\ \sigma_1 \end{bmatrix} = \begin{bmatrix} \alpha^2 \\ 0 \end{bmatrix}$$

Find the error location numbers $\beta_1 = 1/\sigma_1$ and $\beta_2 = 1/\sigma_2$:

$$\text{cofactor} \begin{bmatrix} \alpha^6 & \alpha^0 \\ \alpha^0 & \alpha^0 \end{bmatrix} = \begin{bmatrix} \alpha^0 & \alpha^0 \\ \alpha^0 & \alpha^6 \end{bmatrix} \qquad \det \begin{bmatrix} \alpha^6 & \alpha^0 \\ \alpha^0 & \alpha^0 \end{bmatrix} = \alpha^6 - \alpha^0 = \alpha^2$$

$$\text{Inv} \begin{bmatrix} \alpha^6 & \alpha^0 \\ \alpha^0 & \alpha^0 \end{bmatrix} = \frac{\begin{bmatrix} \alpha^0 & \alpha^0 \\ \alpha^0 & \alpha^6 \end{bmatrix}}{\alpha^2} = \alpha^5 \begin{bmatrix} \alpha^0 & \alpha^0 \\ \alpha^0 & \alpha^6 \end{bmatrix} = \begin{bmatrix} \alpha^5 & \alpha^5 \\ \alpha^5 & \alpha^4 \end{bmatrix}$$

$$\begin{bmatrix} \sigma_2 \\ \sigma_1 \end{bmatrix} = \begin{bmatrix} \alpha^5 & \alpha^5 \\ \alpha^5 & \alpha^4 \end{bmatrix} \begin{bmatrix} \alpha^0 \\ \alpha^2 \end{bmatrix} = \begin{bmatrix} \alpha^4 \\ \alpha^1 \end{bmatrix}$$

From Equations (8.39) and (8.47), we represent $\sigma(X)$ as

$$\sigma(X) = \alpha^0 + \sigma_1 X + \sigma_2 X^2 = \alpha^0 + \alpha^1 X + \alpha^4 X^2$$

We determine the roots of $\sigma(X)$ by exhaustively testing $\sigma(X)$ with each of the field elements. Any element that yields $\sigma(X) = 0$ is a root, and allows us to locate an error.

8.7 (cont'd.)

$$\sigma(\alpha^0) = \alpha^0 + \alpha^1 + \alpha^4 = \alpha^6 \neq \mathbf{0}$$

$$\sigma(\alpha^1) = \alpha^0 + \alpha^2 + \alpha^6 = \mathbf{0} \Rightarrow \textbf{Error}$$

$$\sigma(\alpha^2) = \alpha^0 + \alpha^3 + \alpha^1 = \mathbf{0} \Rightarrow \textbf{Error}$$

$$\sigma(\alpha^3) = \alpha^0 + \alpha^4 + \alpha^3 = \alpha^2 \neq \mathbf{0}$$

$$\sigma(\alpha^4) = \alpha^0 + \alpha^5 + \alpha^5 = \alpha^0 \neq \mathbf{0}$$

$$\sigma(\alpha^5) = \alpha^0 + \alpha^0 + \alpha^0 = \alpha^0 \neq \mathbf{0}$$

$$\sigma(\alpha^6) = \alpha^0 + \alpha^0 + \alpha^2 = \alpha^2 \neq \mathbf{0}$$

$\sigma(\alpha^1) = \mathbf{0}$ indicates the location of an error at $\beta_1 = 1/\sigma_1 = \alpha^6$

$\sigma(\alpha^2) = \mathbf{0}$ indicates the location of an error at $\beta_2 = 1/\sigma_2 = \alpha^5$

(b) Now, we determine the error values e_1 and e_2 associated with locations $\beta_1 = \alpha^6$ and $\beta_2 = \alpha^5$. Any of the four syndrome equations can be used. From Equation (8.38), we use S_1 and S_2.

$$S_1 = \mathbf{r}(\alpha) = e_1\beta_1 + e_2\beta_2$$
$$S_2 = \mathbf{r}(\alpha^2) = e_1\beta_1^2 + e_2\beta_2^2$$

Or, in matrix form: $\begin{bmatrix} \beta_1 & \beta_2 \\ \beta_1^2 & \beta_2^2 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix}$ $\quad \begin{bmatrix} \alpha^6 & \alpha^5 \\ \alpha^5 & \alpha^3 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} \alpha^6 \\ \alpha^5 \end{bmatrix}$

To solve for the error values e_1 and e_2, the above matrix equation is inverted in the usual way, yielding

$$\text{Inv} \begin{bmatrix} \alpha^6 & \alpha^5 \\ \alpha^5 & \alpha^3 \end{bmatrix} = \frac{\begin{bmatrix} \alpha^3 & \alpha^5 \\ \alpha^5 & \alpha^6 \end{bmatrix}}{\alpha^5} = \alpha^2 \begin{bmatrix} \alpha^3 & \alpha^5 \\ \alpha^5 & \alpha^6 \end{bmatrix} = \begin{bmatrix} \alpha^5 & \alpha^0 \\ \alpha^0 & \alpha^1 \end{bmatrix}$$

Now, we solve for the error values, as follows:

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} \alpha^5 & \alpha^0 \\ \alpha^0 & \alpha^1 \end{bmatrix} \begin{bmatrix} \alpha^6 \\ \alpha^0 \end{bmatrix} = \begin{bmatrix} \alpha^4 + \alpha^0 \\ \alpha^6 + \alpha^1 \end{bmatrix} = \begin{bmatrix} \alpha^5 + \alpha^3 \\ \alpha^3 + \alpha^2 \end{bmatrix} = \begin{bmatrix} \alpha^5 \\ \alpha^5 \end{bmatrix}$$

Hence, we show the error polynomial as

$$\mathbf{e}(X) = (111)X^5 + (111)X^6$$
$$= \alpha^5 X^5 + \alpha^5 X^6$$

(c) We correct the flawed codeword from Problem 8.6 by adding the error polynomial to the received polynomial, as follows:

$$\mathbf{U}(X) = \mathbf{r}(X) + \mathbf{e}(X)$$

$$\mathbf{r}(X) = \alpha^3 + \alpha^2 X + \alpha^1 X^2 + \alpha^1 X^3 + \alpha^3 X^4 + \alpha^6 X^5 + \alpha^2 X^6$$

$$\mathbf{e}(X) = \qquad\qquad\qquad\qquad\qquad\qquad + \alpha^5 X^5 + \alpha^5 X^6$$

$$\mathbf{U}(X) = \alpha^3 + \alpha^2 X + \alpha^1 X^2 + \alpha^1 X^3 + \alpha^3 X^4 + \alpha^1 X^5 + \alpha^3 X^6$$

8.7 (cont'd)

Autoregressive modeling technique represent an important way to solve the Reed-Solomon decoding problem. More can be learned about such methods by reading about the Peterson-Gorenstein-Zierler algorithm in reference [5], Blahut, R. E., *Theory and Practice of Error Control Codes*, Addison-Wesley Publishing, Reading, Massachusetts, 1983.

8.8 BLOCK INTERLEAVER:

OUTPUT →

INPUT ↓

```
1 0 0 1
0 1 0 1
1 1 1 0
1 0 0 0
```

OUTPUT SEQUENCE = 1001 0101 1110 1000

CONVOLUTIONAL INTERLEAVER:

INPUT ↓ ↙ OUTPUT

```
X X X 1 0 0 1 X X X
X X X 0 1 0 1 X X X
X X X 1 1 1 0 X X X
X X X 1 0 0 0 X X X
```

The convolutional interleaver in Figure 6.26, can be viewed as: filling the columns of a block interleaver, and then reading the symbols out diagonally, as shown:

OUTPUT SEQUENCE = 1XXX 00XX 011X 1011X 110XX 00XXX0

8.9 (a) (127, 36) code with $d_{min} = 31$.
Therefore, $t_{max} = 15$.

N COLUMNS

M ROWS | INTERLEAVER

bN error burst in channel results in a burst of no more than $\lceil b \rceil$ symbol errors out of the deinterleaver. Each output burst is separated by at least $M - \lfloor b \rfloor$ symbols.

Channel symbol rate = 19.2 kbits/s. Burst of 250 ms results in 4800 symbol errors, $\lceil b \rceil N = 4800$.
A code block of 127 bits can correct 15 errors.
Thus, let $b = 15$; $bN = 4800$; $N = \frac{4800}{b} = \underline{320}$.

$$M - b = 127 \; ; \quad M = 127 + 15 = 142$$

Therefore, a block interleaver of dimensions (142 × 320) will suffice.

<u>End-to-end delay:</u>

For the 142 × 320 interleaver

$$\text{delay} \cong 2MN = \frac{2 \times 142 \times 320 \text{ symbols}}{19.2 \times 10^3 \text{ symbols/s}} = 4.8\text{s}$$

Thus, the interleaver meets the delay requirement.

(b) Burst of 20 ms results in 384 symbol errors
$\lceil b \rceil N = 384$. A sequence of 21 bits can be detected so that 3 errors are corrected. Thus, let $b = 3$.

$$bN = 384; \quad N = \frac{384}{3} = \underline{128}.$$

Each output burst is separated by at least $M - \lfloor b \rfloor$ symbols

$$M - b = 21; \quad M = 21 + 3 = \underline{24}$$

Thus, as a first try consider a (24×128) block interleaver.

End-to-end delay $\cong 2MN = \frac{2 \times 24 \times 128 \text{ symb}}{19.2 \times 10^3 \text{ symb/s}}$

$= 320$ ms.

Thus, to meet the delay requirement, we can choose a <u>convolutional</u> interleaver of size (24×128) with half the delay, or 160 ms.

8.10 (a) $P_E \cong \frac{1}{2^m - 1} \sum_{j=t+1}^{2^m - 1} j \binom{2^m - 1}{j} p^j (1-p)^{2^m - 1 - j}$

$n = 2^{m-1} = 255$. For the CD system, the code is shortened so that in pass 1 of the decoding process $n_1 = 32$, and in pass 2, $n_2 = 28$.

<u>Pass #1</u>: $p_1 = 10^{-3}$

$$P_E \cong \frac{j}{n_1} \binom{n_1}{j} p_1^j (1-p_1)^{n_1 - j}$$

$$P_E \cong \frac{3}{32}\binom{32}{3}(10^{-3})^3(1-10^{-3})^{29}$$

$$= 4.5 \times 10^{-7}$$

<u>Pass #2</u> : $p_2 = 4.5 \times 10^{-7}$; $n_2 = 28$

$$P_E \cong \frac{3}{28}\binom{28}{3}(4.5\times 10^{-7})^3(1-4.5\times 10^{-7})^{25}$$

$$= 3.2 \times 10^{-17}$$

(b) <u>Pass #1</u> : $p_1 = 10^{-2}$

$$P_E \cong \frac{3}{32}\binom{32}{3}(10^{-2})^3(1-10^{-2})^{29}$$

$$= 3.6 \times 10^{-4}$$

<u>Pass #2</u>: $p_2 = 3.6\times 10^{-4}$

$$P_E \cong \frac{3}{28}\binom{28}{3}(3.6\times 10^{-4})^3(1-3.6\times 10^{-4})^{25}$$

$$= 1.6 \times 10^{-8}$$

8.11

a) The likelihood ratios for the received signal are calculated as:

$p(x_k|d_k = +1) = (1/\sigma\sqrt{2\pi})\ \exp(-0.5[(x_k - 1)]/\sigma)^2)$

$p(x_k|d_k = -1) = (1/\sigma\sqrt{2\pi})\ \exp(-0.5[(x_k + 1)]/\sigma)^2)$

Since $x_k = 0.11$ and $\sigma = 1.0$, we compute

$p(x_k|d_k = +1) = (1/\sqrt{2\pi})\ \exp(-0.5[0.11 - 1]^2) = 0.27$

$p(x_k|d_k = -1) = (1/\sqrt{2\pi})\ \exp(-0.5[0.11 + 1]^2) = 0.22$

b) For equiprobable signals, the MAP decision is the same as the maximum likelihood decision, which is that d_k is equal to +1 since

$p(x_k|d_k = +1) > p(x_k|d_k = -1)$.

c) Calculate $p(x_k|d_k = +1)\ P(d_k = +1)$ and $p(x_k|d_k = -1)\ P(d_k = -1)$

$p(x_k|d_k = +1)\ P(d_k = +1) = (0.27)(0.3) = 0.08$, and

$p(x_k|d_k = -1)\ P(d_k = -1) = (0.22)(1.0 - 0.3) = 0.15$

Since $p(x_k|d_k = -1)\ P(d_k = -1) > p(x_k|d_k = +1)\ P(d_k = +1)$ the MAP decision rule of Equation (8.64) is that d_k is equal to −1.

d) Using Equation (8.66), we calculate

$$L(d|x) = \log_e\left(\frac{0.08}{0.15}\right) = \log_e(0.533) = -0.63$$

8.12

The channel measurements yield the following for the LLR values

$$L_c(x_k) = 1.5,\ 0.1,\ 0.2,\ 0.3,\ 2.5,\ 6.0$$

The soft output $L(d_i)$ for the received signal corresponding to data d_i is:

$$L(d_i) = L_c(x_i) + L(d_i) + \{[\ L_c(x_j) + L\ (d_j)\ ] \boxplus L_c(x_{ij})\}$$

And we can write the following for the horizontal and vertical calculations:

$$L_{eh}(d_1) = [L_c(x_2) + L(d_2)] \boxplus L_c(x_{12})$$

$$L_{ev}(d_1) = [L_c(x_3) + L(d_3)] \boxplus L_c(x_{13})$$

$$L_{eh}(d_2) = [L_c(x_1) + L(d_1)] \boxplus L_c(x_{12})$$

$$L_{ev}(d_2) = [L_c(x_4) + L(d_4)] \boxplus L_c(x_{24})$$

$$L_{eh}(d_3) = [L_c(x_4) + L(d_4)] \boxplus L_c(x_{34})$$

$$L_{ev}(d_3) = [L_c(x_1) + L(d_1)] \boxplus L_c(x_{13})$$

$$L_{eh}(d_4) = [L_c(x_3) + L(d_3)] \boxplus L_c(x_{34})$$

$$L_{ev}(d_4) = [L_c(x_2) + L(d_2)] \boxplus L_c(x_{24})$$

Using the approximate relationship in Equation (8.73), we calculate the L_{eh} values first with the fact that $L_c(x_{34}) = L_c(x_{24}) = 0$ since these parity bits are not transmitted. The $L(d)$ are also initially set to zero. Calculating the L_{eh} values yields

$$L_{eh}(d_1) = (0.1 + 0) \boxplus 2.5 = -0.1 \text{ new } L(d_1)$$

$$L_{eh}(d_2) = (1.5 + 0) \boxplus 2.5 = -1.5 \text{ new } L(d_2)$$

$$L_{eh}(d_3) = (0.3 + 0) \boxplus 0 = 0 \text{ new } L(d_3)$$

$$L_{eh}(d_4) = (0.2 + 0) \boxplus 0 = 0 \text{ new } L(d_4)$$

Calculating the L_{ev} values yields

$$L_{ev}(d_1) = (0.2 + 0) \boxplus 6.0 = -0.2 \text{ new } L(d_1)$$

$$L_{ev}(d_2) = (0.3 + 0) \boxplus 0 = 0 \text{ new } L(d_2)$$

$$L_{ev}(d_3) = (1.5 - 0.1) \boxplus 6.0 = -1.4 \text{ new } L(d_3)$$

$$L_{ev}(d_4) = (0.1 - 1.5) \boxplus 0 = 0 \text{ new } L(d_4)$$

Calculating the second iteration of the L_{eh} values yields

$$L_{eh}(d_1) = (0.1 + 0) \boxplus 2.5 = -0.1 \text{ new } L(d_1)$$

$$L_{eh}(d_2) = (1.5 - 0.2) \boxplus 2.5 = -1.3 \text{ new } L(d_2)$$

$$L_{eh}(d_3) = (0.3 + 0) \boxplus 0 = 0 \text{ new } L(d_3)$$

$$L_{eh}(d_4) = (0.2 - 1.4) \boxplus 0 = 0 \text{ new } L(d_4)$$

Calculating the second iteration of the L_{ev} values yields

$$L_{ev}(d_1) = (0.2 + 0) \boxplus 6.0 = -0.2 \text{ new } L(d_1)$$

$$L_{ev}(d_2) = (0.3 + 0) \boxplus 0 = 0 \text{ new } L(d_2)$$

$$L_{ev}(d_3) = (1.5 - 0.1) \boxplus 6.0 = -1.4 \text{ new } L(d_3)$$

$$L_{ev}(d_4) = (0.1 - 1.3) \boxplus 0 = 0 \text{ new } L(d_4)$$

We notice that in this case, due to the puncturing the values of L_{ev} after the second iteration are equal to the values of L_{ev} after the first iteration. Therefore further iterations will not give any further improvements in performance. The soft-output likelihood values are calculated as:

$$L(d) = L_c(x) + L_{eh}(d) + L_{ev}(d)$$

Thus we have

$$L(d_1) = 1.5 - 0.1 - 0.2 = 1.2$$
$$L(d_2) = 0.1 - 1.3 + 0 = -1.2$$
$$L(d_3) = 0.2 + 0 - 1.4 = -1.2$$
$$L(d_4) = 0.3 + 0 + 0 = 0.3$$

Using the MAP decision rule of Equation (8.111), the decoder decides +1 −1 −1 +1 for the transmitted sequence, which is correct. Without coding, two of the four data bits would have been in error.

8.13

a) The output parity sequence is given by 0, 1, 0, 0, 1, 0, 1, 1, 1, 1. In this example, the encoder was not forced back to the all-zeros state, so there are no tail bits.

b) Tthe input sequence is interleaved according to the pattern given. For the given input sequence and interleaving pattern, the interleaved sequence is: 0, 0, 1, 1, 0, 0, 1, 1, 0, 1.

We next encode this sequence which gives us the following output parity sequence: 0, 0, 1, 0, 0, 1, 0, 0, 1, 1.

c) Given the two parity sequences obtained in parts a) and b), and the puncturing pattern, we obtain the parity sequence for the overall codeword. It is: 0, 0, 0, 0, 1, 1, 1, 0, 1, 1.

For the given transmitted data of: 0, 1, 1, 0, 0, 1, 0, 1, 1, 0, we get: Total weight = Weight of data sequence + Weight of parity sequence = 5 + 5 = 10

d) Since the encoders are left unterminated, we need to change the conditions of initialization of the reverse state metrics. The reverse state metrics for the end of the block are all set to the same value, say 1 – instead of using the value 1 only for the all-zeros state and the value 0 for all other states. Also the a priori probability of the last branch metrics in the trellis are all set to 0.5 since there is no a priori information available.

8.14

a) Although the generator polynomial is the same for both component codes, the minimum distance is different since for the first component code, data and parity bits $\{u_k, v_{1k}\}$ are transmitted while for the second component code, only the parity bits $\{v_{2k}\}$ are transmitted, and we do not transmit the interleaved data bits. The minimum distance is generated by considering an input sequence with weight-1 (000…0001000…000). Regardless of the choice of interleaver, an input sequence with weight 1 will always appear at the input of the second encoder. For the encoder shown in Figure P8.1, the component codes have a minimum distance of 3 and 2. Therefore the overall code will have a minimum distance equal to 3 + 2 = 5.

b) For the encoder shown in Figure 8.26, the component codes have a recursive form. If we input an infinitely long weight-1 sequence into the component code, the encoded output will be

given by (000…0001110110110…110…). Thus for a weight-1 input sequence, the output codeword will have infinite weight. The minimum weight output codeword for the recursive code is in fact obtained when the input is given by the weight-3 input sequence (000…000111000…000). For the weight-3 input, the output is given by the following (000…000101000…000) which has a weight of 2. However, when the weight-3 input is interleaved, it is highly likely that the sequence of 3 consecutive 1's will be broken up. Thus, the second encoder is unlikely to produce another output codeword with minimum weight. All we can say about the minimum distance of the output codeword is that it will have a value greater than the minimum weight of $\{u, v_1, v_2\}$ $= 3 + 2 + 2 = 7$.

c) For the case where the weight-2 sequence (00…00100100…00) is input to the encoder shown in Figure 8.26, the output of the encoder is given by (00…00111100…00). This output sequence is said to be self terminating since it does not rely on tail bits in order to force the encoder back to the all-zeros state. If the interleaver fails to break up the (00…00100100…00) sequence, then the output codeword from the second component encoder will also be of the form (00…00111100…00). The resulting output weight is given by 2 + 2(4) = 10.

d) For the case where the weight-2 sequence (00…0010100…00) is input to the encoder shown in Figure 8.26, the output of the encoder is given by (00…0011010110110110111…). This output sequence is not self-terminating as 1's will be produced in the parity output until the encoder is forced back to the all-zeros state at the end of the block. Thus the output of the two encoders potentially have a very large weight if the interleaver fails to break up the (00…0010100…00) sequence. Parts c) and d) illustrate an important aspect of turbo codes in that the inteleaver may be used

to map input sequences which produce low-weight outputs to other input sequences which produce high weight outputs. Thus, when the outputs from data and parity streams are combined, output codewords having a relative high weight may potentially be produced.

8.15

The branch metrics are calculated using equation (8.140). We assume that $A_k = 1$ for all k, and also that the a priori value for π_k^i is 0.5 for all k. The states 00, 10, 01, and 11 are represented by the letters a, b, c, and d, respectively.

a) Using the trellis structure shown in Figure 8.25b, we calculate all the branch metrics at time $k = 1$, that are needed for using the MAP algorithm.

$$\delta_1^{0,a} = (1)(0.5)\exp\{(1/1.3)[(1.9)(-1) + (0.7)(-1)]\} = 0.07$$
$$\delta_1^{1,a} = (1)(0.5)\exp\{(1/1.3)[(1.9)(1) + (0.7)(1)]\} = 3.69$$

The encoder starts in state a at time $k = 1$. Therefore, we assume the values of alpha are all equal to 0 except for state a for which alpha is set equal to 1. We only need the above $\delta_1^{i,m}$ values here. The other six will not be needed, since $\alpha_1^b = \alpha_1^c = \alpha_1^d = 0$. We repeat the branch metric calculations for time $k = 2$.

$$\delta_2^{0,a} = (1)(0.5)\exp\{(1/1.3)[(-0.4)(-1) + (0.8)(-1)]\} = 0.37$$
$$\delta_2^{1,a} = (1)(0.5)\exp\{(1/1.3)[(-0.4)(1) + (0.8)(1)]\} = 0.68$$
$$\delta_2^{0,b} = (1)(0.5)\exp\{(1/1.3)[(-0.4)(-1) + (0.8)(1)]\} = 1.26$$
$$\delta_2^{1,b} = (1)(0.5)\exp\{(1/1.3)[(-0.4)(1) + (0.8)(-1)]\} = 0.20$$

b) We only need the above $\delta_2^{i,m}$ values here. The other four will not be needed, since $\alpha_2^c = \alpha_2^d = 0$.

We have the following initial conditions:

$$\alpha_1^{\,a} = 1 \text{ for } k = 1;$$
$$\alpha_1^{\,b} = \alpha_1^{\,c} = \alpha_1^{\,d} = 0 \text{ for } k = 1$$

From the trellis diagram and Equation (8.131), we obtain the following values for alpha at $k = 2$:

$$\alpha_2^{\,a} = \alpha_1^{\,a}\,\delta_1^{\,0,a} + \alpha_1^{\,c}\,\delta_1^{\,1,c} = (1)(0.07) = 0.07$$
$$\alpha_2^{\,b} = \alpha_1^{\,c}\,\delta_1^{\,0,c} + \alpha_1^{\,a}\,\delta_1^{\,1,a} = (1)(3.69) = 3.69$$
$$\alpha_2^{\,c} = \alpha_1^{\,d}\,\delta_1^{\,0,d} + \alpha_1^{\,b}\,\delta_1^{\,1,b} = 0$$
$$\alpha_2^{\,d} = \alpha_1^{\,b}\,\delta_1^{\,0,b} + \alpha_1^{\,d}\,\delta_1^{\,1,d} = 0$$

and similarly for $k = 3$:

$$\alpha_3^{\,a} = \alpha_2^{\,a}\,\delta_2^{\,0,a} + \alpha_2^{\,c}\,\delta_2^{\,1c} = (0.07)(0.37) = 0.03$$
$$\alpha_3^{\,b} = \alpha_2^{\,c}\,\delta_2^{\,0,c} + \alpha_2^{\,a}\,\delta_2^{\,1,a} = (0.07)(0.68) = 0.05$$
$$\alpha_3^{\,c} = \alpha_2^{\,d}\,\delta_2^{\,0,d} + \alpha_2^{\,b}\,\delta_2^{\,1,b} = (3.69)(0.20) = 0.74$$
$$\alpha_3^{\,d} = \alpha_2^{\,b}\,\delta_2^{\,0,b} + \alpha_2^{\,d}\,\delta_2^{\,1,d} = (3.69)(1.26) = 4.65$$

Note that the $\alpha_3^{\,m}$ values represent the final states at time $k = 3$, and therefore are not used in the computation of the log-likelihood ratio for data bits d_1 and d_2, given below.

c)
$$L(\hat{d}) = \log\left[\frac{\sum_m \alpha_k^m\,\delta_k^{1,m}\,\beta_{k+1}^{f(1,m)}}{\sum_m \alpha_k^m\,\delta_k^{0,m}\,\beta_{k+1}^{f(0,m)}}\right]$$

For $k = 1$:
$$L(\hat{d}_1) = \log_e\frac{(1)(3.69)(2.4)}{(1)(0.07)(4.6)} = 3.31$$

For $k = 2$:
$$L(\hat{d}_2) = \log_e\left[\frac{(0.07)(0.68)(11.5) + (3.69)(0.20)(3.4)}{(0.07)(0.37)(2.1) + (3.69)(1.26)(0.9)}\right] = -0.33$$

We use the MAP decision rule. Since $L(\hat{d}_1) > 0$ and $L(\hat{d}_2) < 0$, then the MAP estimate for the transmitted binary data sequence is $\{1, 0\}$.

8.16

At time $k = 1$, the branch metric is the same as was calculated for Problem 8.15, since both the data bit and the parity bit are transmitted, as in the case of a rate ½ code. In the next interval however, the parity bit is punctured and therefore we only obtain a data bit. This needs to be taken into account when a branch metric is calculated; we ignore the parity-bit component since it does not contribute at all to a branch metric's value in this interval.

At $k = 1$, $\delta_1^{0,a} = 0.07$ and $\delta_1^{1,a} = 3.69$. Only these two $\delta_1^{i,m}$ values are needed here. The other six are not needed, since $\alpha_1^b = \alpha_1^c = \alpha_1^d = 0$.

For time $k = 2$, we only consider the contribution due to the data bit, and we compute:

$$\delta_2^{0,a} = (1)(0.5)\exp[(1/1.3)(-0.4)(-1)] = 0.68$$
$$\delta_2^{1,a} = (1)(0.5)\exp[(1/1.3)(-0.4)(1)] = 0.37$$
$$\delta_2^{0,b} = (1)(0.5)\exp[(1/1.3)(-0.4)(-1)] = 0.68$$
$$\delta_2^{1,b} = (1)(0.5)\exp[(1/1.3)(-0.4)(1)] = 0.37$$

We only need these four $\delta_2^{i,m}$ values here. The other four will not be needed, since $\alpha_1^c = \alpha_1^d = 0$. Based on the above we can calculate our forward state metrics in the usual manner. At time $k = 2$, the forward state metrics will have the same values that they had in the previous problem, but for time $k = 3$, the forward state metrics need to be recalculated based on the new values for the branch metrics.

At time $k = 2$, $\alpha_2^a = 0.07$ and $\alpha_2^b = 3.69$, while $\alpha_2^c = \alpha_2^d = 0$ and at time $k = 3$:

$$\alpha_3^a = \alpha_2^a\,\delta_2^{0,a} + \alpha_2^c\,\delta_2^{1,c} = (0.07)(0.68) = 0.05$$
$$\alpha_3^b = \alpha_2^c\,\delta_2^{0,c} + \alpha_2^a\,\delta_1^{1,a} = (0.07)(0.37) = 0.03$$
$$\alpha_3^c = \alpha_2^d\,\delta_2^{0,d} + \alpha_2^b\,\delta_2^{1,b} = (3.69)(0.37) = 1.37$$
$$\alpha_3^d = \alpha_2^b\,\delta_2^{0,b} + \alpha_2^d\,\delta_2^{1,d} = (3.69)(0.68) = 2.5$$

8.17

The branch metrics are calculated using Equation (8.140). Assume that $A_k = 1$ for all k, and that the a priori value for π_k^i is 0.5 for all k. Using the trellis given in Figure 8.25b, we calculate each of the eight branch metrics at time $k = 1023$ and we repeat this process for those branch metrics that are needed at time $k = 1024$.

For time $k = 1023$:

$$\delta_{1023}^{0,a} = (1)(0.5)\exp\{(1/2.5)[(1.3)(-1) + (-0.8)(-1)]\} = 0.41$$
$$\delta_{1023}^{1,a} = (1)(0.5)\exp\{(1/2.5)[(1.3)(1) + (-0.8)(1)]\} = 0.61$$
$$\delta_{1023}^{0,b} = (1)(0.5)\exp\{(1/2.5)[(1.3)(-1) + (-0.8)(1)]\} = 0.22$$
$$\delta_{1023}^{1,b} = (1)(0.5)\exp\{(1/2.5)[(1.3)(1) + (-0.8)(-1)]\} = 1.16$$
$$\delta_{1023}^{0,c} = (1)(0.5)\exp\{(1/2.5)[(1.3)(-1) + (-0.8)(-1)]\} = 0.41$$
$$\delta_{1023}^{1,c} = (1)(0.5)\exp\{(1/2.5)[(1.3)(1) + (-0.8)(1)]\} = 0.61$$
$$\delta_{1023}^{0,d} = (1)(0.5)\exp\{(1/2.5)[(1.3)(-1) + (-0.8)(1)]\} = 0.22$$
$$\delta_{1023}^{1,d} = (1)(0.5)\exp\{(1/2.5)[(1.3)(1) + (-0.8)(-1)]\} = 1.16$$

For time $k = 1024$, we only need the following two branch metrics:

$$\delta_{1024}^{0,a} = (1)(0.5)\exp\{(1/2.5)[(-1.4)(-1) + (-0.9)(-1)]\} = 1.26$$
$$\delta_{1024}^{1,c} = (1)(0.5)\exp\{(1/2.5)[(-1.4)(1) + (-0.9)(1)]\} = 0.2$$

The encoder ends in state a, so at the terminating time $k = 1025$, we assume the values of the reverse state metrics, β, are all equal to 0 except for state a where β is set equal to 1. The values of β are calculated using equation (8.136). So we have the following initial conditions:

$$\beta_{1025}^{a} = 1$$
$$\beta_{1025}^{b} = \beta_{1025}^{c} = \beta_{1025}^{d} = 0$$

8.17 (cont'd)

From the trellis diagram and Equation (8.136), we obtain the following relationships. For $k = 1024$:

$$\beta_{1024}^{a} = \beta_{1025}^{a}\,\delta_{1024}^{0,a} = (1)(1.26) = 1.26$$
$$\beta_{1024}^{c} = \beta_{1025}^{a}\,\delta_{1024}^{1,c} = (1)(0.2) = 0.2$$

For this example, we do not need to calculate the reverse state metrics for $k = 1023$:

The values of the likelihood ratio are given by the following equation:

$$L(\hat{d}) = \log\left[\frac{\sum_m \alpha_k^m\,\delta_k^{1,m}\,\beta_{k+1}^{f(1,m)}}{\sum_m \alpha_k^m\,\delta_k^{0,m}\,\beta_{k+1}^{f(0,m)}}\right]$$

For $k = 1023$:

$$L(\hat{d}_{1023}) = \log\frac{(7.0)(1.16)(0.2)+(4.2)(0.61)(1.26)}{(6.6)(0.41)(1.26)+(4.0)(0.22)(0.2)} = 0.31$$

For $k = 1024$:

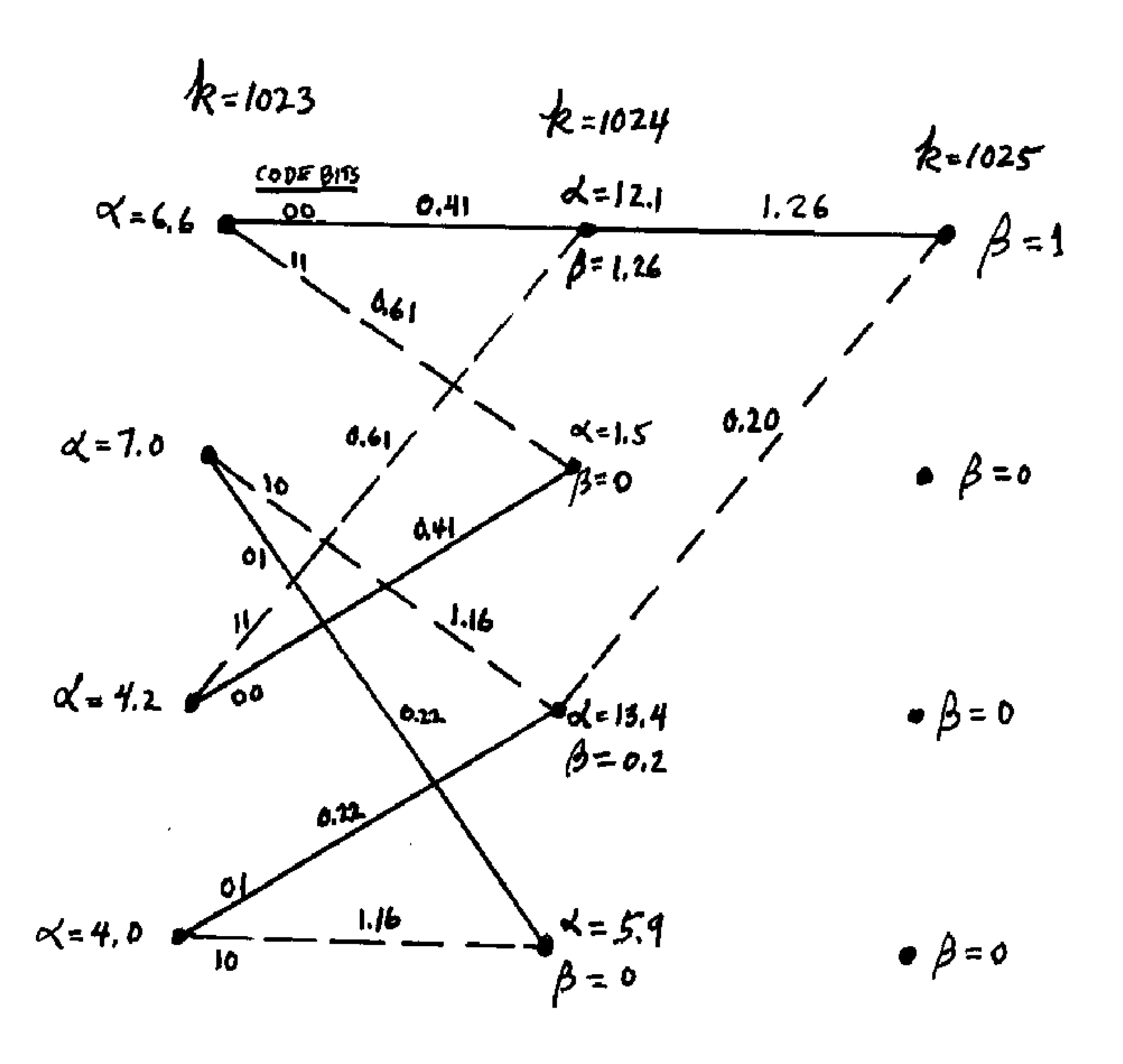
$$L(\hat{d}_{1024}) = \log\frac{(13.4)(0.2)(1)}{(12.1)(1.26)(1)} = -1.74$$

Since $L(\hat{d}_{1023}) > 0$, then we choose data bit 1023 equal to binary 1.

Since $L(\hat{d}_{1024}) < 0$, then we choose data bit 1024 equal to binary 0.

The trellis diagram below (with its metric annotations) can facilitate following the above computations.

8.17 (cont'd.)



k=1023
k=1024
k=1025
CODE BITS
00
0.41
α=12.1
1.26
α=6.6
β=1
11
β=1.26
0.61
α=7.0
0.61
α=1.5
0.20
10
β=0
β=0
0.41
01
11
1.16
α=4.2
00
α=13.4
β=0
0.22
β=0.2
0.22
01
α=4.0
1.16
α=5.9
10
β=0
β=0

8.18

$$L(d|x) = \log\left[\frac{P(d=1|x)}{P(d=-1|x)}\right]$$

For separate observations, x_1 and x_2

$$L(d|x_1, x_2) = \log\left[\frac{P(d=1|x_1, x_2)}{P(d=-1|x_1, x_2)}\right]$$

From Equation (8.67), we can write the log-likelihood ratio (LLR)

$$L(d|x) = \log\left[\frac{P(x|d=1)}{P(x|d=-1)}\right] + \log\left[\frac{P(d=1)}{P(d=-1)}\right]$$
$$= L(x|d) + L(d)$$

Using Bayes' Rule, we observe that

$$P(d=j|x_1, x_2) = \frac{P(d=j, x_1, x_2)}{P(x_1, x_2)} = \frac{P(x_2|x_1, d=j)P(x_1, d=j)}{P(x_1, x_2)}$$
$$= \frac{P(x_2|x_1, d=j)P(x_1|d=j)P(d=j)}{P(x_1, x_2)}$$

If x_1 and x_2 are statistically independent, then we can write

$$P(d=j|x_1, x_2) = \frac{P(x_2|d=j)P(x_1|d=j)P(d=j)}{P(x_1, x_2)}$$

Now we can write the LLR as

$$L(d|x_1, x_2) = \log\left[\frac{P(x_1|d=1)P(x_2|d=1)P(d=1)}{P(x_1|d=-1)P(x_2|d=-1)P(d=-1)}\right]$$

$$= \log\left[\frac{P(x_1|d=1)}{P(x_1|d=-1)}\right] + \log\left[\frac{P(x_2|d=1)}{P(x_2|d=-1)}\right] + \log\left[\frac{P(d=1)}{P(d=-1)}\right]$$

$$= L(x_1|d) + L(x_2|d) + L(d)$$

8.19 (a)

From Equation (8.129)

$$\alpha_k^m = \sum_{m'} \sum_{j=0}^{1} P\left(d_{k-1} = j,\ S_{k-1} = m',\ R_1^{k-2}, R_{k-1} \,\middle|\, S_k = m\right)$$

$$\qquad\qquad A \qquad\quad B \qquad C \quad D \quad E$$

$$P(A,B,C,D|E) = \frac{P(A,B,C,D,E)}{P(E)}$$

$$= \frac{P(C|A,B,D,E)\ P(A,B,D,E)}{P(E)}$$

$$= \frac{P(C|A,B,D,E)\ P(A,B,D|E)\ P(E)}{P(E)}$$

$$\alpha_k^m = \sum_{m'} \sum_{j=0}^{1} P\left(R_1^{k-2} \,\middle|\, S_k = m,\ d_{k-1} = j,\ S_{k-1} = m',\ R_{k-1}\right)$$

$$\times\ P\left(d_{k-1} = j,\ S_{k-1} = m',\ R_{k-1} \,\middle|\, S_k = m\right)$$

(b) Summing over all states *m'* from 0 to $2^{\nu} - 1$ lets us designate $S_{k-1} = b(j, m)$ as the state going backward in time from state *m* via the branch corresponding to an input *j*, yielding Equation (130b).

$$\alpha_k^m = \sum_{j=0}^{1} P\left(R_1^{k-2} \,\middle|\, S_{k-1} = b(j,m)\right)\ P\left(d_{k-1} = j,\ S_{k-1} = b(j,m),\ R_{k-1}\right)$$

Note that $S_{k-1} = b(j, m)$ completely defines the path resulting in $S_k = m$ the current state, given an input j and state m' at the previous time.

8.19 (c)

From Equation (8.133)

$$\beta_k^m = \sum_{m'} \sum_{j=0}^{1} P\left(d_k = j,\ S_{k+1} = m',\ R_k,\ R_{k+1}^N \middle| S_k = m\right)$$

$$\qquad\qquad A \qquad\quad B \qquad C \quad D \qquad E$$

$$P(A,B,C,D|E) = \frac{P(A,B,C,D,E)}{P(E)}$$

$$= \frac{P(D|A,B,C,E)\ P(A,B,C,E)}{P(E)}$$

$$= \frac{P(D|A,B,C,E)\ P(A,B,C|E)\ P(E)}{P(E)}$$

$$\beta_k^m = \sum_{m'} \sum_{j=0}^{1} P\left(R_{k+1}^N \middle| S_k = m,\ d_k = j, S_{k+1} = m',\ R_k\right)$$

$$\times\ P\left(d_k = j,\ S_{k+1} = m',\ R_k \middle| S_k = m\right)$$

$S_k = m$ and $d_k = j$ completely define the path resulting in $S_{k+1} = f\,(j,\ m)$ the next state, given an input j and state m, yielding Equation (8.135):

$$\beta_k^m = \sum_{j=0}^{1} P\left(R_{k+1}^N \middle| S_{k+1} = f(j, m)\right)\ P\left(d_k = j,\ S_k = m,\ R_k\right)$$

8.20

Starting with Equation (8.139)

$$\delta_k^{i,m} = \frac{\pi_k^i}{2^{\upsilon}\sqrt{2\pi}\,\sigma}\,\exp\left[-\frac{1}{2}\left(\frac{x_k - u_k^i}{\sigma}\right)^2\right]dx_k\;\frac{1}{\sqrt{2\pi}\,\sigma}\,\exp\left[-\frac{1}{2}\left(\frac{y_k - v_k^{i,m}}{\sigma}\right)^2\right]dy_k$$

Considering each exponential term separately, we have

$$\exp\left\{-\frac{1}{2\sigma^2}\left[x_k^2 - 2x_k u_k^i + \left(u_k^i\right)^2\right]\right\} = \exp\left\{-\left[\frac{x_k^2 + \left(u_k^i\right)^2}{2\sigma^2}\right]\right\}\,\exp\left\{\frac{2x_k u_k^i}{2\sigma^2}\right\}$$

Similarly for the second exponential, we obtain

$$\exp\left\{-\left[\frac{y_k^2 + \left(v_k^{i,m}\right)^2}{2\sigma^2}\right]\right\}\,\exp\left\{\frac{2y_k v_k^{i,m}}{2\sigma^2}\right\}$$

Then

$$\delta_k^{i,m} = \pi_k^i\,\frac{dx_k dy_k}{2^{\upsilon}2\pi\,\sigma^2}\,\exp\left\{-\left[\frac{x_k^2 + \left(u_k^i\right)^2}{2\sigma^2}\right]\right\}\,\exp\left\{-\left[\frac{y_k^2 + \left(v_k^{i,m}\right)^2}{2\sigma^2}\right]\right\}\exp\left\{\frac{x_k u_k^i + y_k v_k^{i,m}}{\sigma^2}\right\}$$

Observe that since $u_k^i = \pm 1$ and $v_k^{i,m} = \pm 1$, then

$$\delta_k^{i,m} = \frac{dx_k dy_k}{2^{\upsilon}2\pi\,\sigma^2}\,\exp\left\{-\left[\frac{x_k^2 + 1}{2\sigma^2}\right]\right\}\,\exp\left\{-\left[\frac{y_k^2 + 1}{2\sigma^2}\right]\right\}\;\pi_k^i\exp\left\{\frac{x_k u_k^i + y_k v_k^{i,m}}{\sigma^2}\right\}$$

where the first three terms are identified as A_k in Equation (8.140). The A_k term disappears in Equation (8.141a) because in forming $l(\widehat{d}_k)$ it appears in both the numerator and denominator, and hence cancels out.

8.21

Signal processing steps at a receiver must appear in reverse order compared to the way that they were applied at the transmitter. Note, that at the encoder, we first interleave the data bits and then encode them to form parity bits. Thus at the decoder, we must reverse the order by first decoding the parity bits, followed by deinterleaving. If we were to deinterleave first by using a deinterleaver in the lower line (prior to decoding), we would be reversing the necessary order, and the decoder would be faced with parity bits that had been permuted compared to how they were created. Thus the decoding operation would not be successful.

8.22

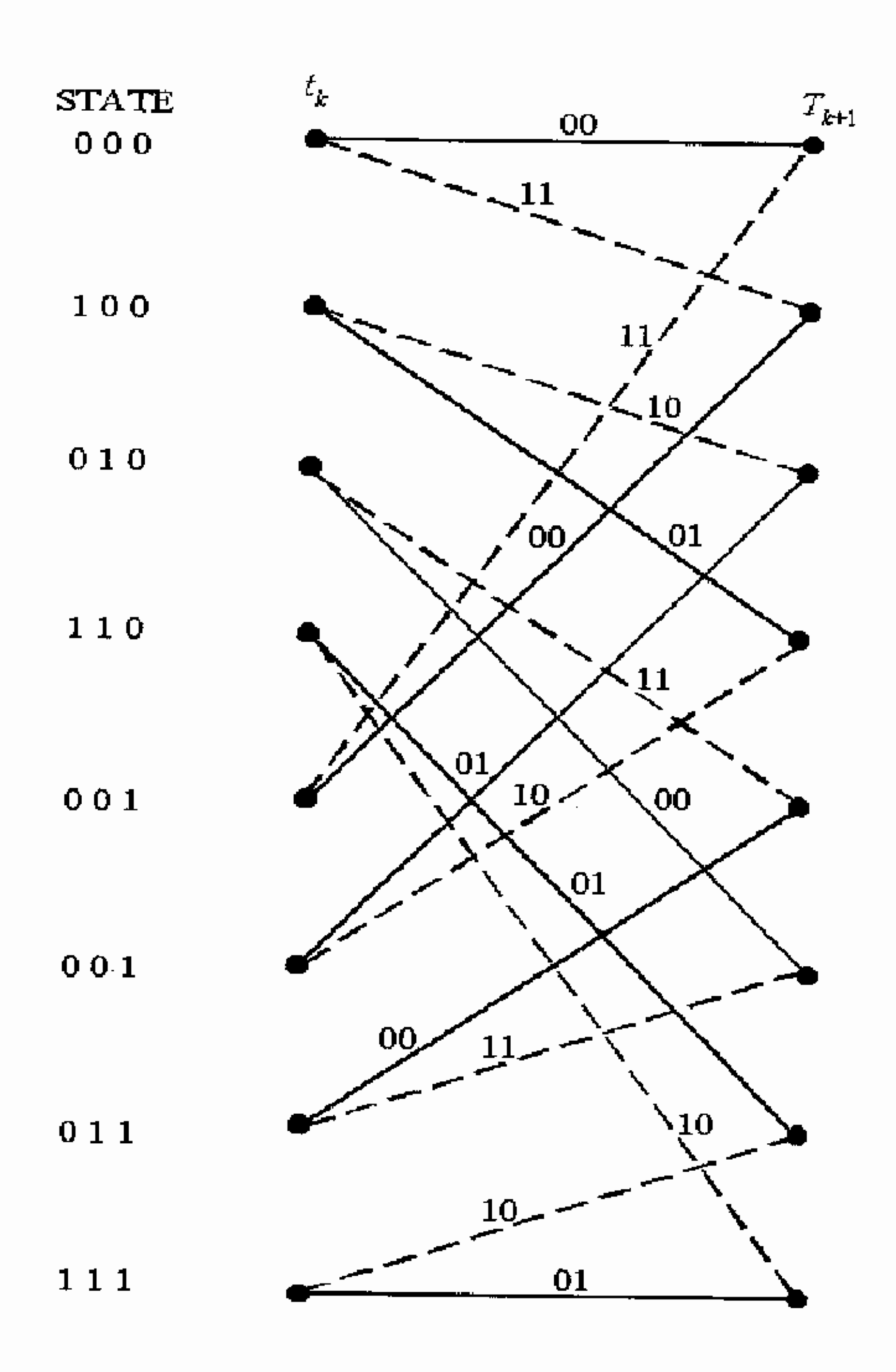
In the Viterbi algorithm, the add-compare-select processor represents a technique that can efficiently yield the maximum likelihood path through a decoding trellis for a given sequence. The maximum a posteriori (MAP) algorithm, unlike the Viterbi algorithm, finds the likelihood ratio for each symbol time interval, and hence can make a MAP decision regarding the symbol during that interval. The MAP algorithm needs to use all of the statistical information associated with the branches of that interval in order to form a likelihood ratio. None of the information can be cast away.

8.23

d_k	a_k	a_{k-1}	a_{k-2}	a_{k-3}	u	v
0	0	0	0	0	0	0
	1	1	0	0	0	1
	1	0	1	0	0	0
	1	0	0	1	0	0
	0	1	1	0	0	1
	0	0	1	1	0	0
	0	1	0	1	0	1
	1	1	1	1	0	1
1	1	0	0	0	1	1
	0	1	0	0	1	0
	0	0	1	0	1	1
	0	0	0	1	1	1
	1	1	1	0	1	0
	1	0	1	1	1	1
	1	1	0	1	1	0
	0	1	1	1	1	0

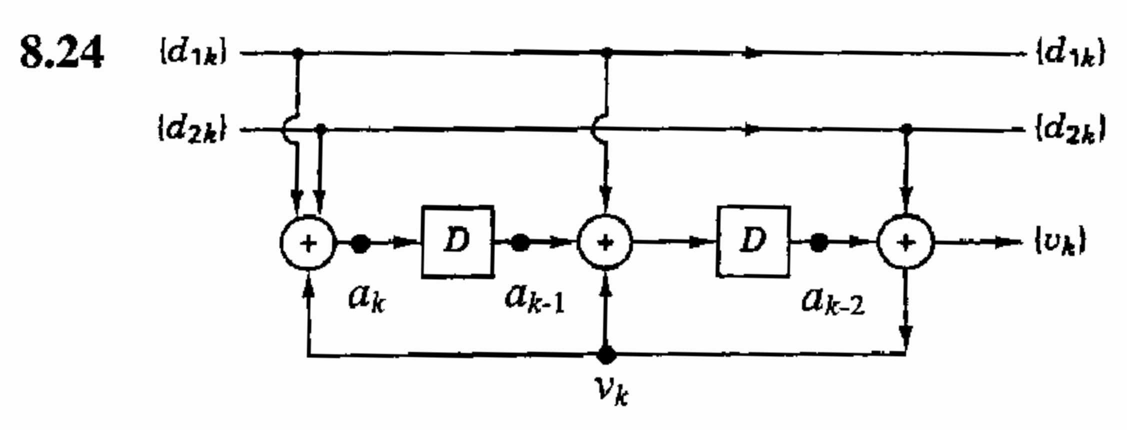
where v is the modulo-2 sum of a_k, a_{k-2}, and a_{k-3}.

8.23 (cont'd.)



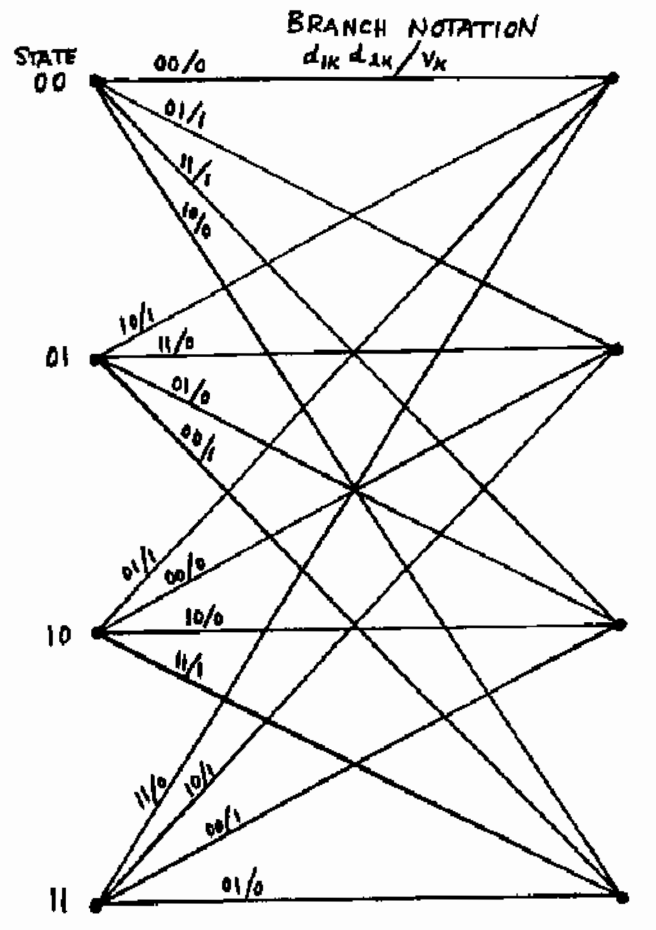
STATE
t_k
T_{k+1}
0 0 0
1 0 0
0 1 0
1 1 0
0 0 1
0 0 1
0 1 1
1 1 1
00
11
11
10
00
01
11
01
10
00
01
00
11
10
10
01

8.24



Starting State		Input bits		Parity output	Current bit	Ending State	
a_{k-1}	a_{k-2}	d_{1k}	d_{2k}	$v_k = a_{k-2} + d_{2k}$	$a_k = d_{1k} + d_{2k} + v_k$	a_k	$a_{k-1} + d_{1k} + v_k$
0	0	0	0	0	0	0	0
		0	1	1	0	0	1
		1	0	0	1	1	1
		1	1	1	1	1	0
0	1	0	0	1	1	1	1
		0	1	0	1	1	0
		1	0	1	0	0	0
		1	1	0	0	0	1
1	0	0	0	0	0	0	1
		0	1	1	0	0	0
		1	0	0	1	1	0
		1	1	1	1	1	1
1	1	0	0	1	1	1	0
		0	1	0	1	1	1
		1	0	1	0	0	1
		1	1	0	0	0	0

8.24 (cont'd.)



Time k	Input bits		Parity output	Current bit	State at Time k		Ending state At time k+1	
	d_{1k}	d_{2k}	$v_k = a_{k-2} + d_{2k}$	$a_k = d_{1k} + d_{2k} + v_k$	a_{k-1}	a_{k-2}	a_k	$a_{k-1} + d_{1k} + v_k$
1	1	1	1	1	0	0	1	0
2	0	0	0	0	1	0	0	1
3	1	1	0	0	0	1	0	1
4	0	0	1	1	0	1	1	1
5	1	1	0	0	1	1	0	0
6					0	0		

Output code-bit sequence (data + parity) is:

111 000 110 001 110

8-25

The likelihood ratio is given by the equation:

$$L(\hat{d}) = \log\left[\frac{\sum_m \alpha_k^m \delta_k^{1,m} \beta_{k+1}^{f(1,m)}}{\sum_m \alpha_k^m \delta_k^{0,m} \beta_{k+1}^{f(0,m)}}\right]$$

$$L(\hat{d}_k) = \log_e\left(\frac{\alpha_k^a \delta_k^{1,a} \beta_{k+1}^{f(1,m)} + \alpha_k^b \delta_k^{1,b} \beta_{k+1}^{f(1,m)} + \alpha_k^c \delta_k^{1,c} \beta_{k+1}^{f(1,m)} + \alpha_k^d \delta_k^{1,d} \beta_{k+1}^{f(1,m)}}{\alpha_k^a \delta_k^{0,a} \beta_{k+1}^{f(0,m)} + \alpha_k^b \delta_k^{0,b} \beta_{k+1}^{f(0,m)} + \alpha_k^c \delta_k^{0,c} \beta_{k+1}^{f(0,m)} + \alpha_k^d \delta_k^{0,d} \beta_{k+1}^{f(0,m)}}\right)$$

We calculate the above values of $L(\hat{d})$ for all k = 6 time intervals. For the four-state code characterized by the trellis of Figure 8.25b, this relationship can be written as:

$$L(\hat{d}_k) = \log\frac{\alpha_k^a \delta_k^{1,a} \beta_{k+1}^b + \alpha_k^b \delta_k^{1,b} \beta_{k+1}^c + \alpha_k^c \delta_k^{1,c} \beta_{k+1}^a + \alpha_k^d \delta_k^{1,d} \beta_{k+1}^d}{\alpha_k^a \delta_k^{0,a} \beta_{k+1}^a + \alpha_k^b \delta_k^{0,b} \beta_{k+1}^d + \alpha_k^c \delta_k^{0,c} \beta_{k+1}^b + \alpha_k^d \delta_k^{0,d} \beta_{k+1}^c}$$

Now, we substitute the given matrix elements into the above equation corresponding to the correct indices. The following values are obtained for the likelihood ratios:

$L(\hat{d}_1)$ = $\log_e(3.60) = 1.28$
$L(\hat{d}_2)$ = $\log_e(0.438) = -0.83$
$L(\hat{d}_3)$ = $\log_e(0.679) = -0.39$
$L(\hat{d}_4)$ = $\log_e(0.476) = -0.85$
$L(\hat{d}_5)$ = $\log_e(0.304) = -1.19$
$L(\hat{d}_6)$ = $\log_e(1.751) = -0.29$

We can therefore estimate from the log-likelihood calculations and the decision rule of Equation (8.111), to decide that the bit was a 1, if $L(\hat{d}_k) > 0$. Otherwise, decide that the bit was a 0. Thus, the MAP decision for the 6 bit sequence is: 1 0 0 0 0 0.

Chapter 9

9.1 (a) $C = W \log_2 (1 + SNR)$

$C = 3000 \log_2 (1001) \cong 30{,}000$ bits/s.

(b) $2^{C/W} = 1 + SNR$

$SNR = 2^{C/W} - 1 = 2^{4800/3000} - 1 = 2.03$
$= 3.08$ dB

(c) $SNR = 2^{19200/3000} - 1 = 83.45 = 19.2$ dB

9.2 $C = W \log_2 (1 + SNR) = 3000 \log_2 (11)$
$= 3000 \times 3.46 = 10{,}380$ bits/s.

Thus, error-free transmission is not possible at 100 kbits/s. Basic trade offs include the reduction of data rate, the increase of channel bandwidth, or improvement in SNR (either through increased power or by the use of low-noise receivers). Also possible is the use of multilevel pulses (to trade-off power for reduced symbol rate), trellis coded modulation.

9.3 $H = \sum_{i=1}^{6} p_i \log_2 (1/p_i)$

$= \frac{1}{2}\log_2 2 + \frac{1}{4}\log_2 4 + \frac{1}{8}\log_2 8 + \frac{1}{16}\log_2 16 + \frac{2}{32}\log_2 32$

$= 1.94$ bits

9.4 $H = \sum_i p_i \log_2 (1/p_i)$

$= 15(0.06)\log_2(1/.06) + 285(0.00035)\log_2 \frac{1}{.00035}$

$= 3.653 + 1.145 = 4.798$

$R_{eff} = 4.798 \times 1000 = 4798$ bits/s.

9.5 (a) $H_{av} = \sum_{i=1}^{16} p_i \log_2 (1/p_i)$

$= 16\left(\frac{1}{16}\log_2 16\right) = 4$ bits/element

$R_{av} = H_{av} \times$ elements/sec

$= 4$ bits/element $\times\ 32 \times 2 \times 10^6$ elements/sec

$= 2.56 \times 10^8$ bits/sec

(b) $H_{av} = 64 \times 16 \left(\frac{1}{64 \times 16}\log_2(16 \times 64)\right) = 10$ bits/elem.

$R_{av} = H_{av} \times 32 \times 2 \times 10^6$ elements/s $= 6.4 \times 10^8$ bits/s.

System capacity required for color is 2.5 times that required for black and white.

(c) $H_{av} = -\sum_i p_i \log_2 p_i$

$$= -100\left(0.003 \log_2 0.003\right) - 300\left(0.001 \log_2 0.001\right)$$

$$- 624\left(0.00064 \log_2 0.00064\right)$$

$$H_{av} = 2,514 + 2,989 + 4,23$$

$$= 9.74 \text{ bits/element}$$

$$R_{av} = H_{av} \times 32 \times 2 \times 10^6 \text{ elements/s.}$$

$$= 6,23 \times 10^8 \text{ bits/s.}$$

9.6 Two source outputs with probabilities p_1 and p_2, such that $p_1 + p_2 = 1$:

$$H = -p_1 \log_2 p_1 - (1-p_1)\log_2(1-p_1)$$

$$\frac{dH}{dp_1} = -\log_2 p_1 + \log_2(1-p_1) = 0$$

$$\log_2 p_1 = \log_2(1-p_1)$$

$$p_1 = 1 - p_1 = p_2$$

For two source outputs, H is maximized when $p_1 = p_2$.

Consider a third source, such that:

$p_1 = p_2$, and $p_1 + p_2 + p_3 = 1$, or

$$2p_1 + p_3 = 1$$

$$H = -\left[2p_1 \log p_1 + (1-2p_1)\log(1-2p_1)\right]$$

$$\frac{dH}{dp_1} = -2\log p_1 + 2\log(1-2p_1) = 0$$

$$p_1 = 1 - 2p_1 = p_3$$

Thus, $p_1 = p_2 = p_3$. By recursion, a system with n source outputs has maximum entropy when each of the outputs have equal probability.

9.7 $H(X|Y) = -\sum_Y P(Y) \sum_X P(X|Y)\log_2 P(X|Y)$

$$= -7\left[(1-P_B)\log_2(1-P_B) + P_B \log_2 P_B\right]$$

$$= -7\left[0.99\log_2 0.99 + 0.01\log_2 0.01\right]$$

$$= 0.566 \text{ bit/character}$$

9.8 (a) Power limited system

Assuming that the usable channel bandwidth can be extended beyond the nominal 2.4 kHz, one can use M-ary FSK modulation, where $M > 2$, and perhaps channel coding as well. The result will be a reduction in required E_b/N_0 and hence the ability to increase the data rate for a fixed level of power (at the expense of bandwidth).

(b) Bandwidth limited system

The described system operates at a system roll-off of $r = 1$. From Equation (3.80), $W = (1+r)\,\frac{R_s}{2}$.

One way of increasing the data rate without increasing bandwidth is to make the channel more bandwidth efficient by reducing the roll-off factor of the filter (steeper skirts).

With a higher data rate, more E_b/N_0 and hence more power is needed, but it is assumed that this is available in the bandwidth limited case. Another suggestion is to use M-ary PSK modulation, thereby increasing the data rate without increasing bandwidth (at the expense of increased E_b/N_0) for a given error performance.

(c) <u>Power and Bandwidth limited systems</u>

If one can provide neither additional power nor bandwidth, one can use Trellis coded modulation to provide a coding gain, and thus permit a greater data rate, without expending additional power or bandwidth.

9.9 $\frac{P_r}{N_0} = \frac{EIRP \; G/T_0}{k L_s L_o}$; $\frac{E_b}{N_0} = \frac{EIRP \; G/T_0}{R \, k \, L_s L_o}$

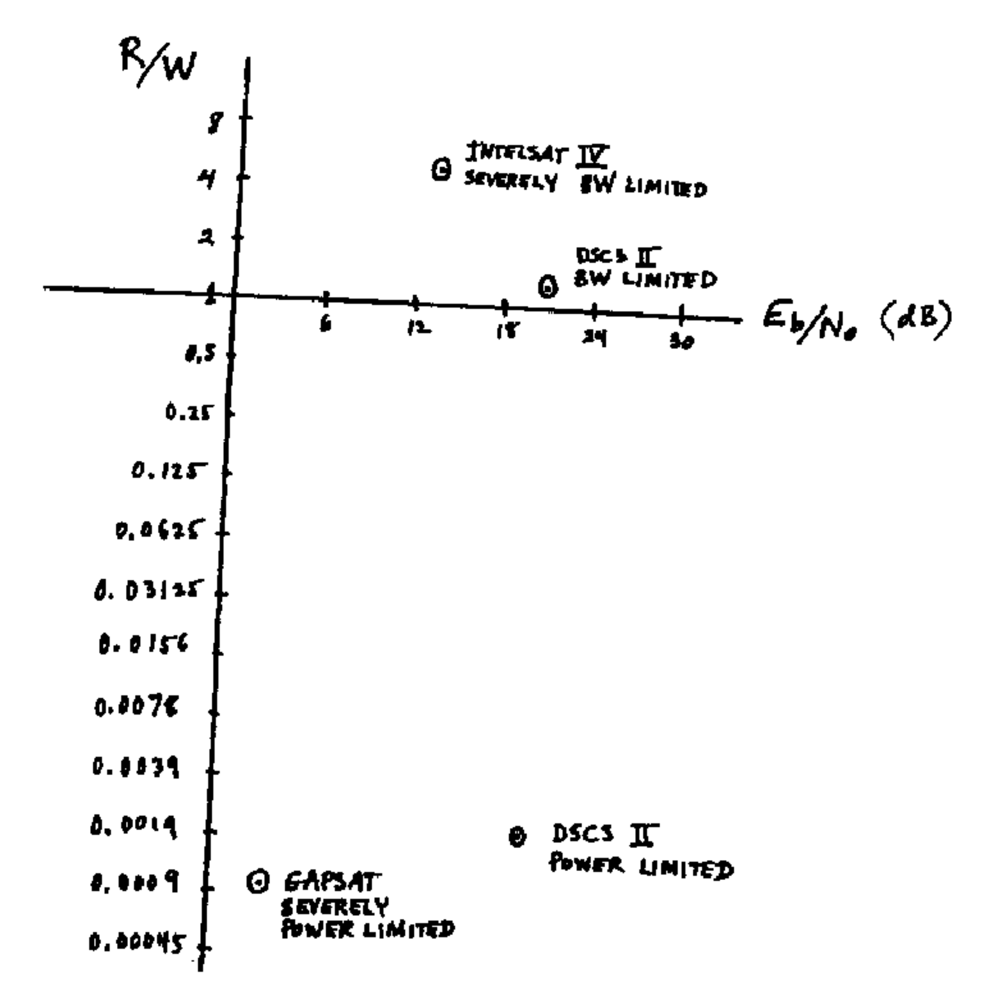
$(E_b/N_0)_{dB} = EIRP + G/T_0 - (R + k + L_s + L_o)$ all in dB

INTELSAT IV: $E_b/N_0 = 13.6$ dB ; $R/W = 4.58$

DSCS II: $E_b/N_0 = 20.6$ dB ; $R/W = 0.002$

DSCS II: $E_b/N_0 = 21$ dB ; $R/W = 1.44$

GAPSAT: $E_b/N_0 = 3.6$ dB ; $R/W = 0.001$



9.10

$R_b = 9600$ bits/s

$$R_s = \frac{R_b}{\log_2 M} = 2400 \text{ symbols/s}$$

which is a reasonable symbol rate choice for a channel with a usable bandwidth of 2400 Hz.

$$\log_2 M = \frac{9600 \text{ bits/s}}{2400 \text{ symbols/s}} = 4 \text{ bits/symbol}$$

$M = 16$; choose 16-ary QAM since a modulation that is bandwidth efficient is called for. From Equation (9.54)

$$P_B = \frac{2(1-L^{-1})}{\log_2 L} Q\left(\sqrt{\left(\frac{3 \log_2 L}{L^2 - 1}\right) \frac{2E_b}{N_0}}\right)$$

where L is the number of amplitude levels in one dimension

$$P_B = \frac{2\left(1 - \frac{1}{4}\right)}{2} Q\left(\sqrt{\left(\frac{3 \times 2}{15}\right) \frac{2E_b}{N_0}}\right); \quad \frac{E_b}{N_0} = 14 \text{ dB} = 25.12$$

$$P_B = \frac{3}{4} Q\left(\sqrt{\frac{4}{5} \frac{E_b}{N_0}}\right) = \frac{3}{4} Q(4.483)$$

$$\cong \frac{3}{4}\left(\frac{1}{4.483\sqrt{2\pi}}\right) \exp\left[-\frac{(4.483)^2}{2}\right]$$

$= 2.89 \times 10^{-6}$ which meets the $P_B \leq 10^{-5}$ requirement.

Therefore, use 16-ary QAM. No coding and no interleaving required.

9.11 Consider noncoherent 8-ary FSK

$$P_E \le \frac{M-1}{2} \exp\left(-\frac{1}{2}\frac{E_s}{N_0}\right) \quad \text{[from Eq. (4.111)]}$$

$$\frac{E_s}{N_0} = \log_2 M \frac{E_b}{N_0} \; ; \quad \frac{E_b}{N_0} = 5.6 \text{ dB} = 3.63$$

$$\frac{E_s}{N_0} = 3 \times 3.63 = 10.89$$

$$P_E \le \frac{7}{2} \exp\left[-\frac{1}{2}(10.89)\right] = 1.5 \times 10^{-2}$$

$$P_B = \left(\frac{2^{k-1}}{2^k - 1}\right) P_E = \frac{4}{7} \times 1.5 \times 10^{-2} = 8.6 \times 10^{-3}$$

This does not meet the $P_B \le 10^{-5}$ requirement.

Bandwidth required without coding

$$W = M\left(\frac{1}{T}\right)$$

$$R_s = \frac{R_b}{\log_2 M} = \frac{9600 \text{ bits/s}}{3} = 3200 \text{ symbols/s}$$

Let $\frac{1}{T} = 3200$ Hz

$$W = 8 \times 3200 \text{ Hz} = 25.6 \text{ kHz}$$

Thus, we can afford to use coding which will expand the required bandwidth. But, we cannot use a rate ½ code since that will expand the bandwidth beyond the available 40 kHz.

Try the (127, 92) BCH code.

Bandwidth required is:

$$W = 25.6 \text{ kHz} \times \frac{127}{92} = 35.3 \text{ kHz}$$

We are O.K. on bandwidth. Now, check the error performance.

The code has $d_{min} = 11$ and therefore can correct $\frac{d_{min} - 1}{2} = 5$ error patterns or fewer anywhere in a block of 127 bits.

$$P_M \cong \binom{127}{6} p_c^6 (1 - p_c)^{121} \quad \text{where } p_c = 8.6 \times 10^{-3}$$

$$P_M \approx \binom{127}{6} (8.6 \times 10^{-3})^6 (1 - 8.6 \times 10^{-3})^{121}$$
$$= 7.35 \times 10^{-4}$$

where P_M is the probability of a message or block error.

$$P_M = 1-(1-P_B)^{127} \; ; \quad P_B = 1-(1-P_M)^{1/127}$$

$P_B = 5.79 \times 10^{-6}$ which meets the requirement of $P_B \le 10^{-5}$. No interleaving is called for.

9.12 Bandwidth efficiency is called for. Therefore, consider the 16-ary QAM modulation. Using Eq. (9.54)

$$P_B = \frac{3}{4} Q\left(\sqrt{\frac{4}{5}\frac{E_b}{N_0}}\right) \; ; \quad \frac{E_b}{N_0} = 8 \text{ dB} = 6.309$$

$P_B = \frac{3}{4} Q(2.247)$. Using Table B.1,

$$P_B = \frac{3}{4} \times 0.0123 = 9.22 \times 10^{-3}$$

which does not meet the $P_B \le 10^{-5}$ requirement. We therefore need coding to improve the error performance. With 16-ary QAM, 9600 bits/s are

transmitted as $\frac{9600 \text{ bits/s}}{4 \text{ bits/symbol}}$ =
2400 symbols/s, which can comfortably utilize a 2400 Hz bandwidth. With the (127, 92) BCH code the 2400 Hz will be expanded to
$2400 \text{ Hz} \times \frac{127}{92} = 3313 \text{ Hz}$ (which meets the 3400 Hz requirement).
The code provides the following error performance:

$$P_M \cong \binom{127}{6}\left(9.22\times10^{-3}\right)^6\left(1-9.22\times10^{-3}\right)^{121}$$
$$= 1.03\times10^{-3}$$

$$P_B = 1-\left(1-P_E\right)^{1/127} = 8.1\times10^{-6}$$

which meets the spec of $P_B \leq 10^{-5}$.
Interleaving is required to handle bursts of $9600 \text{ bits/s} \times \frac{127}{92} = 13{,}252.1$ coded symbols/s for a duration of 100 ms. Therefore the burst to be protected contains approximately 1326 symbols

To select one of the M row × N column interleavers, consider that any bN noise burst can result in no more than $\lceil b \rceil$ symbol errors. Let $b = 5$ since the error-correcting capability of the code will correct all 5 or fewer error patterns in a block of 127 code symbols.

$$bN = 1326 \; ; \quad N = \left\lceil \frac{1326}{5} \right\rceil = 266.$$

Each output error burst is separated by $M - b$ or more symbols. Thus, choose this separation to be equal to the code block size

$$M - b = 127 \; ; \quad M = 127 - 5 = 122$$

Thus the interleaver dimensions should be $M \times N = (122 \times 266)$. Therefore, we must select the (150×300) convolutional interleaver.

9.13 (a)

First, consider using the $(n, k) = (24, 12)$ code. From Equation (9.23), we calculate the received E_b/N_0

$$\frac{E_b}{N_0}(\text{dB}) = \frac{P_r}{N_0}(\text{dB-Hz}) - R(\text{dB-bits/s})$$
$$= 70 \text{ dB-Hz} - 60 \text{ dB-bits/s} = 10 \text{ dB} = 10$$

For the case of 8-PSK modulation and the (24, 12) code we calculate E_c/N_0 and E_s/N_0 as

$$\frac{E_c}{N_0} = \left(\frac{k}{n}\right)\frac{E_b}{N_0} = \left(\frac{12}{24}\right) 10 = 5$$
$$\frac{E_s}{N_0} = (\log_2 M)\frac{E_c}{N_0} = 3 \times 5 = 15$$

Next, we use the approximation in Equation (9.25) yielding

$$P_E(M) \approx 2Q\left[\sqrt{\frac{2E_s}{N_0}}\ \sin\left(\frac{\pi}{m}\right)\right] \text{ for } M > 2$$
$$P_E(8) \approx 2Q\left[\sqrt{30}\ \sin\left(\frac{\pi}{8}\right)\right] = 2Q\ (2.096)$$

Using Table B.1 for $Q(\cdot)$, we get $P_E = 0.0362$.

Assuming Gray coding, Equation (9.27) yields the channel bit-error rate, p_c out of the demodulator

$$p_c \approx \frac{P_E}{\log_2 M} = \frac{0.0362}{3} = 1.2 \times 10^{-2}$$

We enter this value of p_c on the abscissa of Figure 6.21, and for the (24, 12) code transfer function, we can get the decoded bit-error probability, $P_B \doteq 5 \times 10^{-5}$, which is not small enough to meet the required performance. We next consider the other candidate codes.

For the (127, 64) BCH code, we note that the rate of the code is $k/n = 64/127 \approx 1/2$. Hence, repeating the above computation yields approximately the same channel bit-error probability ($p_c \approx 1.18 \times 10^{-2}$) as before. However, in this case of entering p_c on the abscissa of Figure 6.21, we are using the transfer function of the (127, 64) code, which yields a decoded bit-error probability that meets the required $P_B \leq 10^{-7}$.

We now consider the final candidate, the (127, 36) BCH code, and repeat the same computations shown above. (Remember to use the given transfer-function intercepts to guide you in making graphical estimates.) The computations now yield $p_c \doteq 3.8 \times 10^{-2}$. When entering this point on the abscissa of Figure 6.21, and examining the transfer function of the (127, 36) BCH code, we find that $P_B > 10^{-7}$. Hence, of the three candidate codes, only one, the (127, 64) BCH code meets the requirements.

(b) From Figure 6.21 it should be clear that the (127, 36) BCH code is the most capable of the group. Even its label states that $t = 15$, meaning that within a block of 127 bits, this code can connect any combination of 15 or fewer errors. A natural initial guess might be to choose the (127, 36) BCH code—but that would have been incorrect. The reason that the more capable (127, 36) code does not meet our requirements is related to the fact that in a real-time communications system, there are two mechanisms at work: 1) more redundancy makes for more powerful codes and, 2) more redundancy results in less energy per channel bit. As the rate of a code decreases (from 1 to 0), there will be an improvement in P_B due to mechanism 1. But eventually, mechanism 2 results in more errors out of the demodulator and outweighs the benefits of mechanism 1. (See Section 9.7.7.2.)

(c) Using Equations (9.25) and (9.27) we compute the uncoded received E_b/N_0 corresponding to $p_c = 10^{-7}$, as follows:

$$p_c = 10^{-7} \cong \frac{P_E}{\log_2 M} \approx \frac{2Q\left[\sqrt{\frac{2E_s}{N_0}}\ \sin\left(\frac{\pi}{m}\right)\right]}{\log_2 M}$$

$$1.5\times10^{-7} = Q\left[\left(\sqrt{2}\right)(0.3827)\sqrt{\frac{E_s}{N_0}}\right] = Q(x)$$

where $Q(x) \approx \frac{1}{x\sqrt{2\pi}}\exp\left(-x^2/2\right)$

By trial and error: $x = 5.13$. Thus, $\frac{E_s}{N_0} = 89.85$ and $\frac{E_b}{N_0} = \left(\frac{E_s}{N_0}\right)\left(\frac{1}{3}\right)$

and thus, $\left(\frac{E_b}{N_0}\right)_u = 29.2 = 14.8\text{ dB}$. Therefore, the coding gain from Equation (9.32) is: G(dB) = 14.8 – 10 = 4.8 dB.

9.14

Space loss is $L_s = \left(\frac{4\pi d}{\lambda}\right)^2 = 3.94\times10^{15}$ (or 156 dB). System temperature is $T_s = T_A + (LF-1)290\ =\ 290+(20-1)290\ =\ 5800\text{K}\ (\text{or } 37.6\,\text{dB-K})$. Since $(E_b/N_0)_r = M\times(E_b/N_0)_{reqd}$, and margin is 0 dB (or 1), we solve for $\left(E_b/N_0\right)_r$ with the parameters given and the basic link margin equation $M = \frac{\text{EIRP}\ G/T}{(E_b/N_0)_{reqd} R\, k\, L_s\, L_o}$ which yields the value of $\left(E_b/N_0\right)_r$ = 5.2 dB (or 3.31). Since the channel is bandlimited, we choose MPSK modulation. To meet the bandwidth requirement of 3000 Hz, and at the same time conserve power, we choose the smallest M-ary value for MPSK, which is 16-PSK. This modulation (with perfect filtering) will require a symbol rate (and transmission bandwidth of $R/\log_2 M$ = 9600/4 = 2400 Hz. We also need to select a BCH code

such that the 2400 Hz modulation bandwidth is not expanded beyond 3000 Hz. Hence, this places a restriction on the code redundancy *n/k,* and for a (127, *k*) code, the smallest value allowable for *k* is dictated by the fact that 127/*k* must not exceed 3000/2400. Therefore, *k* must be larger than 102, and our choice from Table 9.2 to provide the most redundancy (and still meet our bandwidth constraint) is the (127, 106) triple error correcting code.

The chosen 16-PSK modulation dictates that the available $(E_b/N_0)_r$ will yield an $E_s/N_0 = (\log_2 M)(k/n)(E_b/N_0) = 4\times(106/127)\times 3.31 = 11.05$. We use this value for finding symbol-error probability, P_E, in the following relationship:

$$P_E(M) \approx 2Q\left[\sqrt{\frac{2E_s}{N_0}}\ \sin\left(\frac{\pi}{M}\right)\right]$$

where Q( ) is the complementary error function. With the computed value of $E_s/N_0 = 11.05$, we find that $P_E = 0.359$. Since Gray coding is called for, the probability of channel bit error is $p_c = \frac{P_E}{\log_2 M} = 0.09$. Now, we can calculate the decoded bit-error probability, P_B, using the approximation of Equation (6.46). Note, that if p_c is small (or E_b/N_0 reasonably large) then the first two terms in the summation of Equation (6.46) are usually adequate because of rapid convergence. However, in this problem, convergence occurs after about 25 terms, yielding a decoded $P_B = 0.09$ (There is no coding gain!) If only the first two terms in the summation of Equation (6.46) had been used, the result would give the erroneous appearance of acceptable error performance.

9.15

Since the channel is bandwidth limited, using Table 9.1, we select MPSK with the smallest M possible (for the sake of power conservation). That is, we select 16-PSK, which requires a theoretical Nyquist minimum bandwidth of 2400 Hz and an $E_b/N_0 = 17.5$ dB at $P_B = 10^{-5}$. We compute the received E_b/N_0 using Equation (9.23)

$$\left(\frac{E_b}{N_0}\right)_r \text{ (dB)} = \frac{P_r}{N_0} \text{ (dB-Hz)} - R \text{ (dB-bits/s)}$$

$$= 54.8 - 10\log_{10} 9600 = 54.8 - 39.8 = 15 \text{ dB } = 31.6$$

Thus, for $P_B \le 10^{-5}$, it is necessary to use error-correction coding with a coding gain of 17.5 dB − 15 dB = 2.5 dB. Since we may only expand the bandwidth by a factor of 2700/2400 (12.5% increase), our only code choice is the (127, 113) BCH code shown in Table 9.3. We verify the decoded bit-error probability performance as:

$$\frac{E_s}{N_0} = \left(\log_2 M\right)\left(\frac{k}{n}\right)\frac{E_b}{N_0} = (4)\left(\frac{113}{127}\right)(31.6) = 112.5$$

$$P_E(M) \approx 2Q\left[\sqrt{\frac{2E_s}{N_0}}\ \sin\left(\frac{\pi}{M}\right)\right]$$

For $M = 16$

$$P_E \approx 2Q\left[\sqrt{225.09}\ \ (0.1951)\right] = 2Q(2.9269) = 0.00173$$

The channel bit-error probability out of the demodulator is $p_c \approx \frac{P_E}{4} = 0.0043$ (assuming Gray coding). Then the decoded bit-error probability is found using Equation (9.41)

$$P_B = \frac{1}{127}\sum_{j=t+1=3}^{127} j\binom{127}{j}(0.00043)^j(1-0.00043)^{127-j} \approx 6\times10^{-7}$$

which meets the system requirements.

9.16 (a) MPSK: $P_E(M) \cong 2Q\left[\sqrt{\frac{2E_s}{N_0}}\left(\sin\frac{\pi}{M}\right)\right]$

$$P_B \cong \frac{2}{\log_2 M} Q\left[\sqrt{\frac{2(\log_2 M)E_b}{N_0}}\left(\sin\frac{\pi}{M}\right)\right]$$

[from Equation (3.20)]

and QAM: $P_B \cong \frac{2(1-L^{-1})}{\log_2 L} Q\left[\sqrt{\left(\frac{3\log_2 L}{L^2-1}\right)\frac{2E_b}{N_0}}\right]$

[from Equation (9.54)], where $M = L^2$

The ratio of the average power, as a function of M, for PSK signaling over that for QAM signaling (for a fixed P_B) is approximately:

$$\frac{\left[\frac{1}{\sqrt{2\log_2 M}\left(\sin\frac{\pi}{M}\right)}\right]^2}{\frac{1}{\frac{6\log_2 L}{L^2-1}}}$$

For large M, $\sin(\pi/M) \approx \pi/M$, thus the above ratio becomes:

$$\frac{3 \log_2 M / (M-1)}{2 \log_2 M \left(\pi^2/M^2\right)}$$

$$= \frac{3M^2}{2(M-1)\pi^2} = \frac{\text{average power for MPSK}}{\text{average power for QAM}}$$

(b) For a fixed P_e and an increasing alphabet size in the case of MPSK, average power increases as a function of M^2. However, in the case of QAM average power increases as a linear function of M.

9.17 (a) $\frac{R}{W} = \frac{28.8 \text{ kbits/s}}{3429 \text{ Hz}} \approx 8.4 \text{ bits/s/Hz}$

(b) $C = W \log_2\left(1 + \frac{S}{N}\right) = W \log_2\left(1 + \frac{E_b}{N_0}\frac{R}{W}\right) \qquad \frac{C}{W} = \log_2\left(1 + \frac{E_b}{N_0}\frac{R}{W}\right)$

Let $R = C$: Then, $2^{C/W} = 1 + \left(\frac{E_b}{N_0}\frac{C}{W}\right)$ and $\frac{W\left(2^{C/W} - 1\right)}{C} = \frac{E_b}{N_0} = 10$

$\frac{3429\left(2^{C/3429} - 1\right)}{C} = 10$ By trial and error, $C \approx 20{,}300$ bits/s.

(c) $\frac{E_b}{N_0} = \frac{W\left(2^{C/W} - 1\right)}{C} = \frac{3429}{28{,}800}\left(2^{8.4} - 1\right) = 40.1$ (or ≈ 16 dB)

9.18 (a)

$$r_1 = \frac{0.5}{\sin 36^\circ} = 0.851$$

$$r_2 = \frac{0.5}{\sin\left(\frac{360^\circ}{22}\right)} = 1.775$$

Since $r_2 > 2r_1$, the minimum distance between the two rings is > 1.

(b) Average power for the (5, 11) circular constellation is: $\frac{5}{16} r_1^2 + \frac{11}{16} r_2^2 = \boxed{2.39}$

Average signal power for the 4×14 square constellation is:

b • c
½ 1
a b

point a has amplitude $\sqrt{0.5^2 + 0.5^2}$
power is 0.5

point b has amplitude $\sqrt{0.5^2 + 1.5^2}$
power is 2.5

point c has amplitude $\sqrt{1.5^2 + 1.5^2}$
power is 4.5

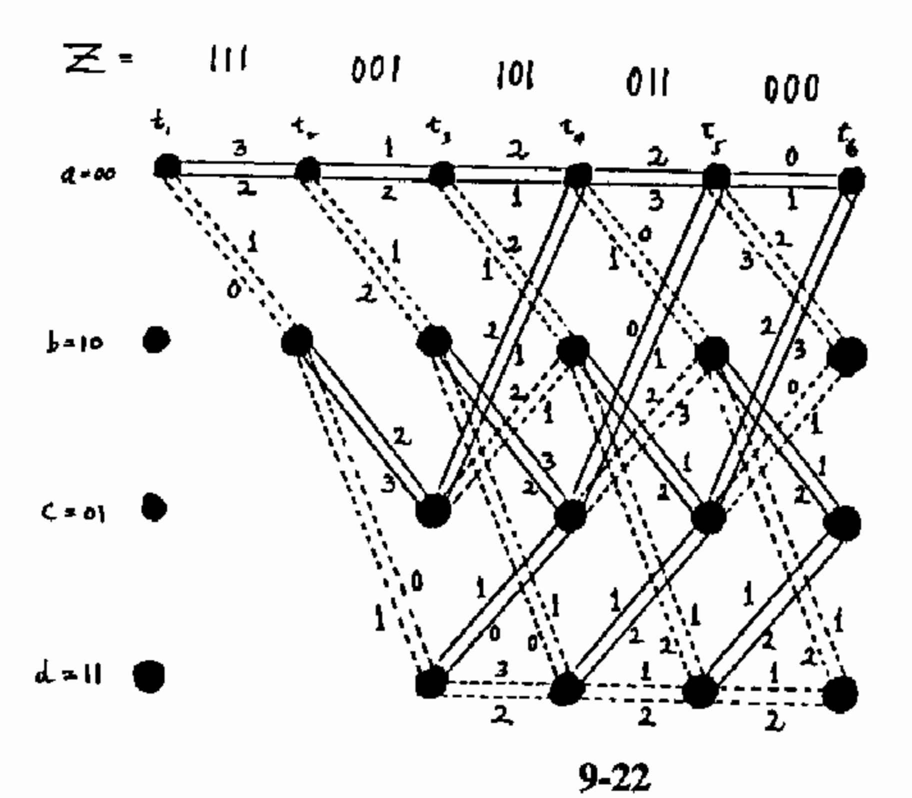
Average power $= \frac{1}{4}(0.5) + \frac{1}{2}(2.5) + \frac{1}{4}(4.5)$

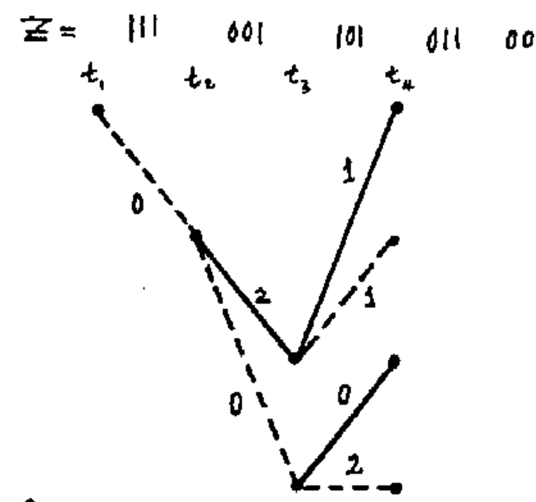
$= \boxed{2.5}$

(c) The square constellation requires only 2 amplitudes and two orthogonal phases, whereas the circular constellation requires either 2 amplitudes and 15 or 16 phases, or numerous amplitudes with two orthogonal phases.

9.19 (a) The encoder trellis illustrates the output branch words corresponding to the encoder state transitions. The leftmost digit is the uncoded digit. A dashed line indicates that the even input bit is a one. A solid line indicates that it is a zero. The decoder trellis shows the Hamming distance difference between each triple group of received output code bits and corresponding branch words from the encoder trellis.

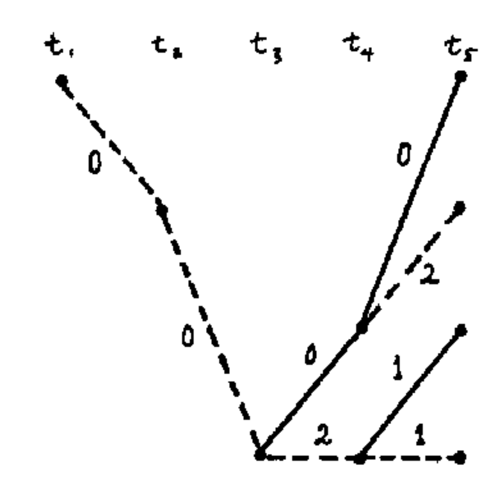
Branchword
u_1 u_2 u_3
t_i t_{i+1}
State a = 00
b = 10
c = 01
d = 11





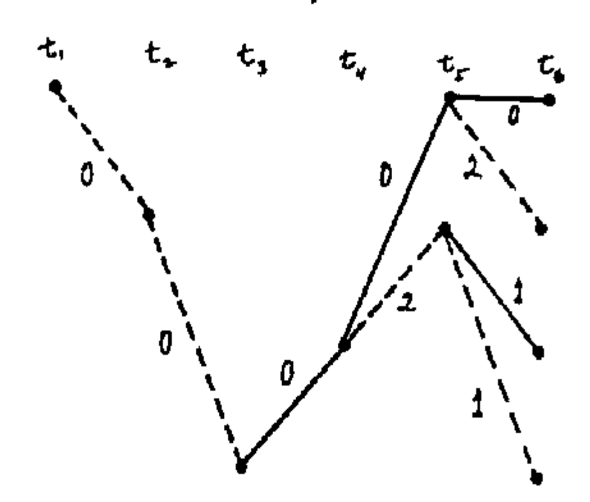
$\hat{m}$ = 11 First two decoded information bits.

At time t_4, merging paths are pruned by eliminating the larger metric paths, and the first pair of bits are decoded.



$\hat{m}$ = 11 01 First four decoded information bits.

At time t_5, again eliminating the larger metric paths results in an additional pair of decoded bits.



$\hat{m}$ = 11 01 10 First 6 decoded information bits

Finally, at time t_6, after again eliminating the larger metric paths, we see that the first 6 decoded information bits are $\hat{m}$ = 110110.

(b) To determine if any channel bits in Z had been inverted by noise on the channel, we simply encode the sequence $\hat{m} = 110110$ found in part (a), using the encoder trellis. The resulting codeword U is identical to Z. We therefore conclude that none of the bits had been inverted by the channel.

(c) If the channel had been specified as Gaussian instead of BSC, we would have used Euclidean distances, similar to those shown in Figures 9.31 and 9.32, for the decoding procedure. The elimination of larger metric paths would proceed in the same way as in the case of Hamming distances.

9.20

For this problem, the circuit, trellis, and Ungerboeck partitioning diagrams are seen in Figures 9.29, 9.30, and 9.32, respectively. The error-event path with the minimum distance is seen in Figure 9.24 as the darkened path labeled with waveform numbers 2, 1, 2. The minimum distance-squared $d_f^2 = 36$ as is shown in Equation (9.61). Note that in this example, the parallel paths do not characterize the error-event with the minimum distance. Using Equation (9.63), the average power for the signal waveforms is $S_{av} = 21$. In this example, the reference waveform set was chosen such that $d_{ref}^2 = 4$, and the average power for this reference set is $S'_{av} = \dfrac{1^2 + 16^2}{2} = 128.5$.

Coding gain is computed using Equation (9.62) as follows:

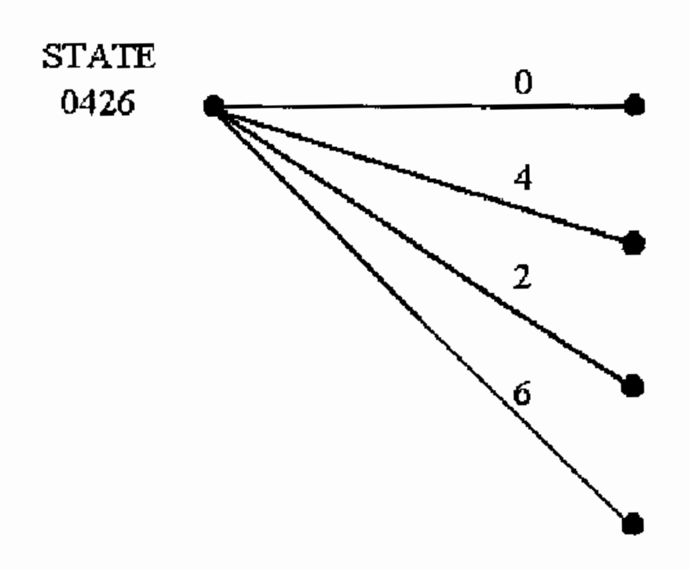
$$G(\text{dB}) = 10 \log_{10} \left(\frac{d_f^2 / S_{av}}{d_{ref}^2 / S'_{av}} \right) = 10 \log_{10} \left(\frac{36/21}{4/128.5} \right) = 17.4 \text{ dB}$$

Of course this answer seems to violate the Shannon prediction of coding gain. Would anyone use such a paradoxical reference set as the one given here? Absolutely not. However, sometimes the choice of a reference set involves judgement. Generally, any reasonable choice for a reference set yields similar coding gains. But, this problem purposely starts with a very unreasonable reference choice to emphasize that the resulting 17.4 dB coding gain reflects two different mechanisms: 1) the improvement due to coding, and 2) the improvement due to the better signal-waveform set (compared to the sub-optimum reference set).

Because trellis-coded modulation involves coding in conjunction with modulation, the "so called" coding gain can be made to appear arbitrarily large by simply starting with a reference set that is sufficiently poor.

9.21

We draw the trellis diagram, and we assign waveforms to trellis branches according to the Ungerboeck assignment rules. We then label each state according to the waveforms assigned to the branches that emanate (top to bottom) from that state. For example, the first state and its branches are labeled as follows:

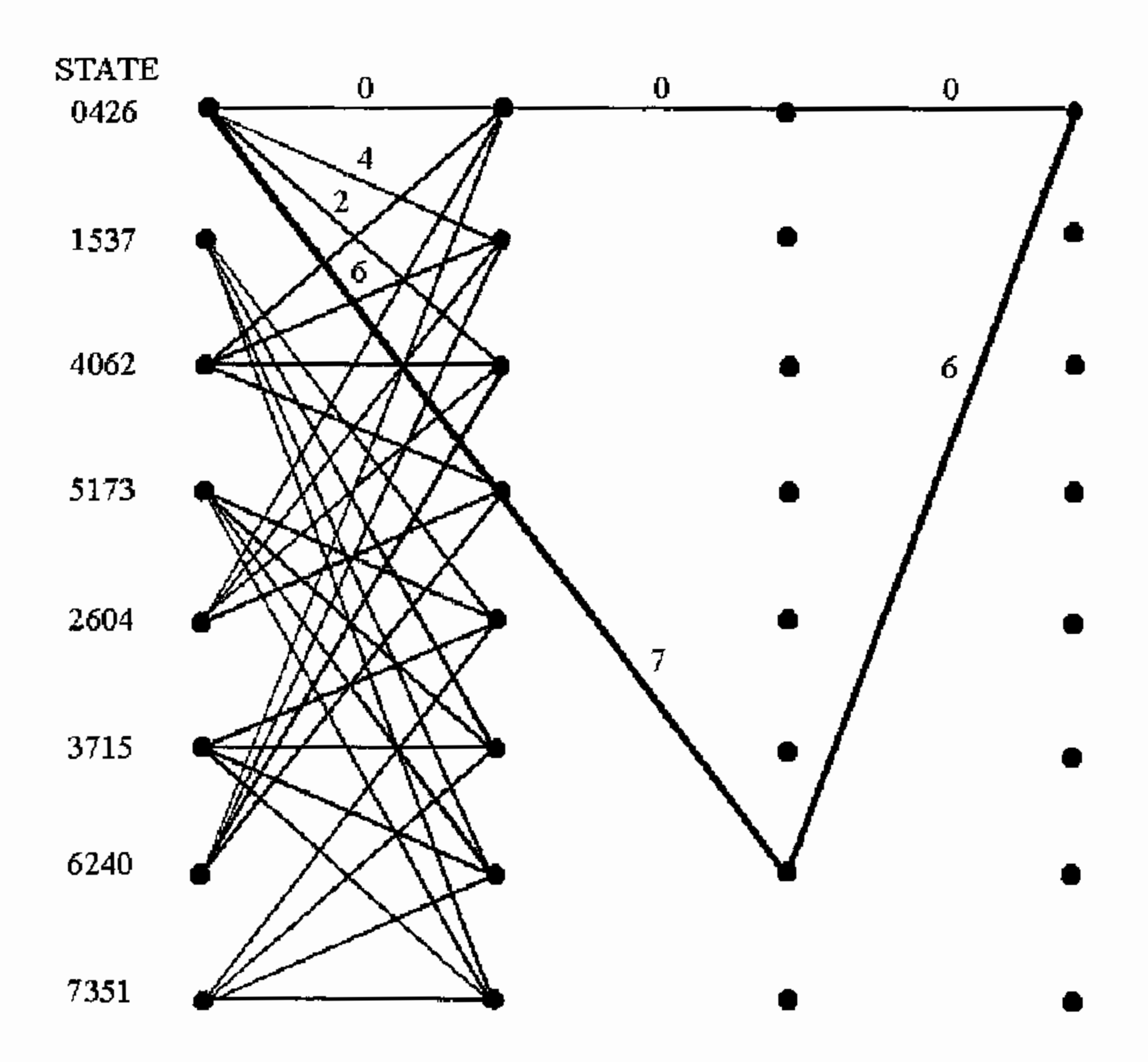


After drawing and labeling three sections of the eight-state machine, a methodical search for the error-event path with the minimum distance yields the darkened path shown below.

We find d_f^2 using Figure 9.22 as

$$d_f^{\,2} = d_1^{\,2} + d_0^{\,2} + d_1^{\,2} = 2 + 0.585 + 2 = 4.585$$

For the 4-PSK reference set, we find $d_{\text{ref}}^2 = 2$ from Figure 9.23.



The asymptotic coding gain relative to a 4-PSK reference set is found using Equation (9.62) as follows:

$$G(\text{dB}) = 10 \log_{10} \left(\frac{d_f^{\,2} / S_{\text{av}}}{d_{\text{ref}}^2 S'_{\text{av}}} \right)$$

The average power is unity for both the signal waveform set and the reference waveform set. Therefore, $G(\text{dB}) = 10 \log_{10} \left(\frac{4.585}{2} \right) = 3.6 \text{ dB}$.

Chapter 10

10.1 The solution is based on the use of the final value theorem as it is applied to the Fourier transform of phase error — Equation (10.9). Thus, for frequency lock

$$\lim_{j\omega \to 0} \frac{(j\omega)^2 \, \Theta_i(\omega)}{j\omega + K_o F(\omega)} = E \text{ (finite constant)}$$

The input time-varying phase is derived from the problem statement to be:

$$\theta(t) = \int_0^t \omega_D(x)\, dx + \theta_o$$

$$= \frac{\omega_o}{c} \int_0^t v(x)\, dx + \theta_o$$

$$= \frac{\omega_o}{c} \left[v(t) - v(0) \right] + \theta_o$$

$$= +\left(\frac{\omega_o}{c}\right) D \cos mt + \theta_o$$

It is seen that the boundary condition $\theta(0) = 0$ is satisfied by $\theta_o = -\frac{\omega_o D}{c}$.

Thus, $\theta(t) = \frac{\omega_o D}{c}\left[\cos(mt) - 1\right]$.

The Fourier transform of this input phase is:

$$\Theta(\omega) = \frac{m^2}{j\omega(\omega^2 - m^2)}\left(\frac{\omega_o D}{c}\right)$$

Thus,

$$\lim_{j\omega \to 0} \frac{(j\omega)^2 \Theta(\omega)}{j\omega + K_o F(\omega)}\left(\frac{\omega_o D}{c}\right)$$

$$= \lim_{j\omega \to 0} \frac{j\omega m^2}{(\omega^2 - m^2) K_o F(\omega)}\left(\frac{\omega_o D}{c}\right)$$

Thus, the lowest order loop filter that will allow the right-hand-side of this equation to be finite is the all-pass $F(\omega) = 1$. Therefore, the lowest order loop is first order.

<u>10.2</u> From Equation 7), the Fourier transform of the phase error is given by:

$$E(\omega) = \frac{j\omega \Theta(\omega)}{j\omega + K_o F(\omega)}$$

From Problem 10.1,

$$\Theta(\omega) = \left(\frac{\omega_o D}{c}\right) \frac{m^2}{j\omega(\omega^2 - m^2)}$$

$$\therefore E(\omega) = \left(\frac{\omega_o D}{c}\right) \frac{m^2}{(\omega^2 - m^2)(j\omega + K_o F(\omega))}$$

For the all-pass $F(\omega) = 1$:

$$E_{ap}(\omega) = \left(\frac{W_0 D}{c}\right) \frac{m^2}{(\omega^2 - m^2)(jW + K_0)}$$

The inverse Fourier transform yields the all-pass case:

$$e_{ap}(t) = \frac{m^2\left(\frac{W_0 D}{c}\right)}{K_0^2 + m^2}\left[\cos(mt) - e^{-K_0 t} - \frac{K_0}{m} \sin(mt)\right]$$

For the low-pass case $\left[F(\omega) = \frac{\omega_1}{j\omega + \omega_1} \right]$

$$E_{lp}(\omega) = \left(\frac{W_0 D}{c}\right) \frac{m^2(j\omega + \omega_1)}{(\omega^2 - m^2)(-\omega^2 + j\omega\omega_1 + K_0\omega_1)}$$

The inverse Fourier transform yields:

$$e_{lp}(t) = \left(\frac{W_0 D}{c}\right) m^2 \Big[A \sin(mt + \varphi_1) + B \exp(-\omega_1 t/2) \sin(Gt + \varphi_2) \Big]$$

Where:

$$A = \frac{1}{m}\sqrt{\frac{\omega_1^2 + m^2}{(K_0\omega_1 - m^2)^2 + m^2(\omega_1)^2}}$$

$$B = \sqrt{\frac{K_0 \omega_1}{G^2[(K_0\omega_1 - m^2)^2 + m^2(\omega_1)^2}}$$

$$G = \sqrt{K_0\omega_1 - (\omega_1)^2/4}$$

$$\Phi_1 = \tan^{-1}\left(\frac{m}{\omega_1}\right) - \tan^{-1}\left(\frac{\omega_1 m}{K_0\omega_1 - m^2}\right)$$

and $\varphi_2 = \tan^{-1}\left(\frac{2G}{\omega_1}\right) - \tan^{-1}\left(\frac{\omega_1 G}{G^2 - m^2}\right)$

The point is not to be concerned about obtaining the exact form of these results, but to show that the error is a linearly decreasing function of K_0 (for large K_0), and thus, the assumption of the linearized equations will be appropriate for K_0 sufficiently large.

10.3 The expression for acceleration, $a(t) = At^2$, implies the expression for velocity $v(t) = \frac{A}{3}t^3 + V_0$, where V_0 is the initial value of relative velocity. The Doppler frequency shift caused by this relative value is given by:

$$\Delta\omega_D(t) = \frac{\omega_0 v(t)}{c} = \omega_0\left(\frac{A}{3}t^3 + V_0\right)/c$$

The corresponding expression for the time-varying phase shift will be:

$$\varphi(t) = \int \Delta\omega_0(t)\,dt = \omega_0\left(\frac{A}{12}t^4 + V_0 t + d_0\right)/c$$

Since the problem states that the receiver is initially in phase lock, $\varphi(0) = \omega_0 d_0/c = 0 \Rightarrow d_0 = 0$.

The Fourier transform of $\varphi(t)$ is given by: $\Theta(j\omega) = \omega_0\left[\frac{2A}{(j\omega)^5} + \frac{V_0}{(j\omega)^2}\right]/c$

From Equation (10.9),

$$\lim_{t\to\infty} e(t) = \lim_{j\omega\to 0} \frac{(j\omega)^2\,\Theta(\omega)}{j\omega + K_0 F(\omega)}$$

$$= \lim_{j\omega\to 0} \frac{\omega_0\left[\frac{2A}{(j\omega)^3} + V_0\right]/c}{j\omega + K_0 F(\omega)}$$

In order for the right-hand-side of the above expression to be finite, the filter transfer function $F(\omega)$ must have terms in the denominator of the order of $(j\omega)^3$ or larger. In order for the right-hand-side of the relation to be

zero, there must be terms of the order of $(j\omega)^4$ or larger. Thus, in order to maintain frequency lock, the loop must be at least 4th order (which is a <u>very</u> high order loop for practical applications), and in order to have a chance at maintaining phase lock, the loop must be at least 5th order.

<u>10.4</u> The loop bandwidth is given by Equation (10.30) as:

$$2B_L = \frac{1}{2\pi}\int_{-\infty}^{\infty} |H(j\omega)|^2 d\omega$$

where from Equation 6)

$$H(j\omega) = \frac{K_o F(\omega)}{j\omega + K_o F(\omega)}$$

For a first-order loop, $F(\omega) = 1$ (all-pass). Thus,

$$H(j\omega) = \frac{K_o}{j\omega + K_o} \Longrightarrow |H(j\omega)|^2 = \frac{K_o^2}{\omega^2 + K_o^2}$$

$$\therefore\ 2B_L = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{K_o^2}{\omega^2+K_o^2}\, d\omega = \frac{K_o}{2\pi}\int_{-\infty}^{\infty} \frac{dx}{x^2+1}\, dx$$

$$= \left[\frac{K_o}{2\pi}\ \arctan(x)\right]_{-\infty}^{\infty} = \frac{K_o}{2\pi}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = \frac{K_o}{2}$$

Therefore, $B_L = K_o/4$

10.5 As with problem 10.4,

$$2B_L = \frac{1}{2\pi}\int_{-\infty}^{\infty} |H(j\omega)|^2\, d\omega$$

where $H(j\omega) = \dfrac{K_o F(\omega)}{j\omega + K_o F(\omega)}$; $F(\omega) = \dfrac{\omega_1}{j\omega + \omega_1}$

Then, $|H(j\omega)|^2 = H(j\omega)\,H(-j\omega)$

$$= \frac{K_o^2 F(\omega) F(-\omega)}{(j\omega + K_o F(\omega))(-j\omega + K_o F(-\omega))}$$

$$= \frac{K_o^2 \omega_1^2}{\omega^4 + (\omega_1^2 - 2K_o\omega_1)\omega^2 + K_o^2\omega_1^2}$$

Under the assumption that $K_o \geq \omega_1$ the integral relation in the 'hint' is valid, and plugging in and

manipulating will yield the desired result.

$$q = \sqrt[4]{\frac{a}{c}} = \sqrt{K_0 \omega_1}$$

$$\cos h = \frac{-b}{2\sqrt{ac}} = \frac{2K_0 - \omega_1}{2K_0}$$

$$\Rightarrow \cos\left(\frac{h}{2}\right) = \sqrt{\frac{4K_0 - \omega_1}{4K_0}} \; ; \; \sin(h) = \frac{\sqrt{4K_0\omega_1 - \omega_1^2}}{2K_0}$$

$$\Rightarrow 2B_L = K_0/4$$

Therefore, $B_L = K_0/8$

10.6 One cycle slip per day is one cycle slip per 86,400 seconds.

From Equation (10.38),

$$86400 \le \pi \exp(2\rho)/4B_L$$

and $\rho = \frac{1}{\sigma_\theta^2} = 1/2N_0B_L$. For a first-order loop (ref: problem 10.4)

$$B_L = \frac{K_0}{4} \Rightarrow \rho = 2/K_0N_0$$

Thus, $86400 \le \pi \exp(4/K_o N_o)/K_o$

$$\ln\left(\frac{86400\,K_o}{\pi}\right) \le 4/K_o N_o$$

$$N_o \le \frac{4}{K_o \ln\left(\frac{86400\,K_o}{\pi}\right)} \quad \text{for } \frac{86400\,K_o}{\pi} \ge 1$$

10.7 A probability density function is a non-negative function whose integral over its range is unity. The function $p(\varphi)$ can be seen to be non-negative by inspection. Also, from the integral form for the zeroth order modified Bessel function of the first kind,

$$I_o(x) = \frac{1}{\pi}\int_0^{\pi} \exp(x\cos\theta)\,d\theta,$$

the integral of $p(\varphi)$ is unity when integrated over its range $(-\pi, \pi)$. The mean of $p(\varphi)$ is zero, as can be seen from the fact that:

$$m = \int_{-\pi}^{\pi} \varphi \exp(\rho\cos\varphi) \big/ 2\pi I_0(\rho)\, d\varphi$$

and $\exp(\rho\cos\varphi)$ is an even function of φ, while φ itself is an odd function. The variance,

$$\sigma^2 = \int_{-\pi}^{\pi} \varphi^2 \exp(\rho\cos\varphi) \big/ 2\pi\, I_0(\rho)\, d\varphi$$

does not appear to have a closed form solution. A formulation in terms of an infinite summation is available through the expansion

$$\exp(\rho\cos\varphi) = I_0(\rho) + 2\sum_{k=1}^{\infty} I_k(\rho)\cos(k\varphi)$$

which provides the answer

$$\sigma^2 = \frac{1}{I_0(\rho)}\left[\frac{\pi^2}{3} I_0(\rho) + 4\sum_{k=1}^{\infty} (-1)^k \frac{I_k(\rho)}{k^2}\right]$$

<u>10.8</u> Given the distribution

$p(T) = 1 - \exp(-T/T_m)$, the density function is: $\dot{p}(T) = \dfrac{\exp(-T/T_m)}{T_m}$

Then the mean is given by

$$\overline{T} = \int_0^{\infty} T\, p(T)\, dT = \int_0^{\infty} \frac{T}{T_m} \exp(-T/T_m)\, dT$$

$= T_m$ (after change of variables and integration by parts).

The second moment is:

$$\overline{T^2} = \int_0^{\infty} (T^2/T_m) \exp(-T/T_m)\, dT$$

$= 2T_m^2$ (after a change in variable and integration by parts).

$$\therefore \sigma_T^2 = \overline{T^2} - (\overline{T})^2 = T_m^2$$

Less than 1 hour apart would be

$$p(1/24) = 1 - \exp(-1/24) = 0.041$$

More than 3 days would be

$$1 - p(3) = \exp(-3) = 0.050$$

10.9 From Equation (10.52), the maximum sweep rate is given by:

$$\Delta\dot{\omega} \cong \frac{1}{2}\omega_n^2 (1 - 2\sigma_\theta).$$

where $\sigma_\theta^2 = 2N_0 B_L$; $2B_L = \frac{1}{2\pi}\int_{-\infty}^{\infty} |H(j\omega)|^2 d\omega$

From Equation (10.6)

$$H(j\omega) = \frac{K_0 F(\omega)}{j\omega + K_0 F(\omega)} \quad ; \quad F(\omega) = \frac{\omega_1}{j\omega + \omega_1}$$

$$= \frac{K_0 \omega_1}{(j\omega)^2 + \omega_1 (j\omega) + K_0 \omega_1}$$

$$= \frac{1}{\left(\frac{j\omega}{\sqrt{K_0 \omega_1}}\right)^2 + \sqrt{\frac{\omega_1}{K_0}}\left(\frac{j\omega}{\sqrt{K_0 \omega_1}}\right) + 1}$$

By identification with Equation (10.51)

$$\omega_n = \sqrt{K_0 \omega_1} = K_0/\sqrt{2}$$

and from Equation (10.31) $\sigma_\theta^2 = 2N_0 B_L$.

From Problem 10.5, $B_L = K_0/8$

$$\therefore 1000 \text{ rad/s} \cong \frac{1}{2}\left(\frac{K_0}{\sqrt{2}}\right)^2\left(1 - 2(2N_0 K_0/8)\right)$$

$$= \frac{K_0^2}{4}\left(1 - \frac{N_0 K_0}{2}\right)$$

$$N_0 = \frac{8}{K_0^3}\left(\frac{K_0^2}{4} - 1000\right)$$

To find largest value of N_o that can be accomodated, determine K_o for $dN_o/dK_o = 0$ and evaluate in above expression.

$$N_o = \frac{2}{K_o} - \frac{8000}{K_o^3}$$

$$\frac{dN_o}{dK_o} = \frac{-2}{K_o^2} + \frac{24000}{K_o^4} = 0$$

$$\frac{24000}{K_o^2} = 2 \; ; \quad K_o = \sqrt{12000} = 109.5$$

$$\left(N_o\right)_{max} = \frac{2}{K_o} - \frac{8000}{K_o^3}\bigg|_{K_o = 109.5} = 0.0122$$

<u>10.10</u> From Equation (10.54)

$$|\bar{\epsilon}|/T \cong 0.33/\sqrt{KE_b/N_o}$$

where $0.1/T \cong 1/KT$

$\Rightarrow K = 10, \quad E_b/N_o = 10\,dB = 10$

$\Rightarrow |\bar{\epsilon}|/T \cong 0.033$

From Equation (10.55)

$$\sigma_\epsilon/T \cong 0.411/\sqrt{KE_b/N_o} = 0.0411$$

$$\Rightarrow \sigma_\epsilon^2/T^2 = (0.0411)^2 = 0.00169$$

Chebyshev's inequality states

$$\text{Prob}\left(|x-\bar{x}| \geq \epsilon\right) \leq \sigma^2/\epsilon^2$$

Thus, $\text{Prob}\left(|x-\bar{x}| \geq 3\bar{x}\right) \leq \sigma^2/9\bar{x}^2$

$$= \frac{0.00169}{9(0.033)^2} = 0.172$$

<u>10.11</u> The minimum header will allow for no errors. Therefore, from Equation (10.84), $P_{FA} = 1/2^N$.

There are 3.1536×10^7 seconds in a year, therefore 3.1536×10^9 bits per year.

$$\therefore \; P_{FA} = \frac{1}{3.1536 \times 10^9}$$

$\therefore 2^N = 3.1536 \times 10^9 \Rightarrow N = 31.55$

or a 32 bit header.

The probability of missing this header is the probability that there are errors in the 32 bits

$$\therefore \; P_m = \sum_{j=1}^{32} \binom{32}{j} p^j (1-p)^{32-j} = 1-(1-p)^{32}$$

For a channel bit error probability of 10^{-5}

$$P_m = 1-(1-10^{-5})^{32} = 3.2\times 10^{-4}$$

For a channel bit error probability of 2×10^{-2}

$$P_m = 1-(1-0.02)^{32} = 0.476$$

For two or fewer errors:

$$P_{FA} = \left[\binom{N}{2}+\binom{N}{1}+\binom{N}{0}\right]\Big/2^N = \frac{1}{3.1536\times 10^9}$$

$$\left[\frac{N(N-1)}{2}+N+1\right]\Big/2^N = \frac{N^2+N+2}{2^{N+1}} = 3.17\times 10^{-10}$$

Solving by iteration, $N \cong 42$ bits

$$P_m = \sum_{j=3}^{42}\binom{42}{j}p^j(1-p)^{42-j} = 1-(0.98)^{42}$$

$$-\,0.02\times 42(0.98)^{41} - 0.0001\times 42\times 41\,(0.98)^{40}$$

$$= 0.128$$

10.12 The desired center frequency is the nominal transmission frequency modified by the expected Doppler shift

$$\Delta f = (15000\ \text{m/s})(8\ \text{GHz})/3\times 10^8\ \text{m/s}$$
$$= 400\ \text{kHz}$$

Since the space probe is receding,

Center frequency = $8 GHz - \Delta f = 7.9996$ GHz

The bandwidth is determined by the combination of velocity uncertainty (Doppler uncertainty) and reference drift.

$$\text{Doppler uncertainty} = (3\ m/s)(8\ GHz)/3\times10^{8}\ m/s = 80\ Hz$$

From Equation (8.64), the drift of the probe's frequency reference is:

$$\Delta\omega(T) = (8\ GHz)(10^{-9}\ Hz/Hz/day)(30\ days) = 240\ Hz$$

The ground station frequency-reference drift is:

$$\Delta\omega(t) = (8\ GHz)(10^{-13}\ Hz/Hz/day)(30\ days) = 0.024\ Hz$$

Since the uncertainties can be either positive or negative,

$$\text{Bandwidth} = 2\,(80 + 240 + 0.024) = 640\ Hz$$

The time of arrival uncertainty is a combination of the range uncertainty and the effects of oscillator drift.

Time uncertainty due to range is:

$$\Delta t_r = (3\ m/s)(86\,400\ sec/day)(30\ days)/3\times10^{8}\ m/s = 25.9\ ms$$

From Equation (10.90), the uncertainty due to oscillator drift is:

$$\Delta t(T) = \frac{1}{2}(10^{-9})(30)^2 = 4.5 \times 10^{-7} \text{ day}$$

$$= (4.5 \times 10^{-7})(86400 \text{ sec/day}) = 38.9 \text{ ms}$$

The time uncertainty is the sum of the terms Δt_p and $\Delta t(T)$ = $25.9 + 38.9$ ms

$$= 64.8 \text{ ms}$$

10.13 The combined drift must be less than 1 kHz/day. From Equation (10.89), $\Delta\omega(T) = \omega_0 \delta T = 1 \text{ kHz}$

where $\omega_0 = 10$ GHz, and $T = 1$ day

$$\delta = \frac{10^3}{10^{10}} = 10^{-7} \text{ Hz/Hz/day}$$

Therefore, the drift rate of the individual reference must be

$$\leq 5 \times 10^{-8} \text{ Hz/Hz/day}$$

This will require a high quality crystal oscillator, as a minimum.

10.14 (a)

$$e = e_o + \left(\frac{f_e}{f_r}\right)t + \frac{1}{2}\,a\,t^2$$

$$0 = \frac{-4 \times 10^{-3}\ \text{SEC}}{86{,}400\ \text{SEC/DAY}} + 0 + \frac{1}{2}\,(2 \times 10^{-10}/\text{DAY})\ t^2$$

$$t^2 = \frac{4.6296 \times 10^{-8}\ \text{DAY}}{10^{-10}/\text{DAY}} = 462.96\ \text{DAY}^2$$

$$t = 21.5\ \text{DAYS}$$

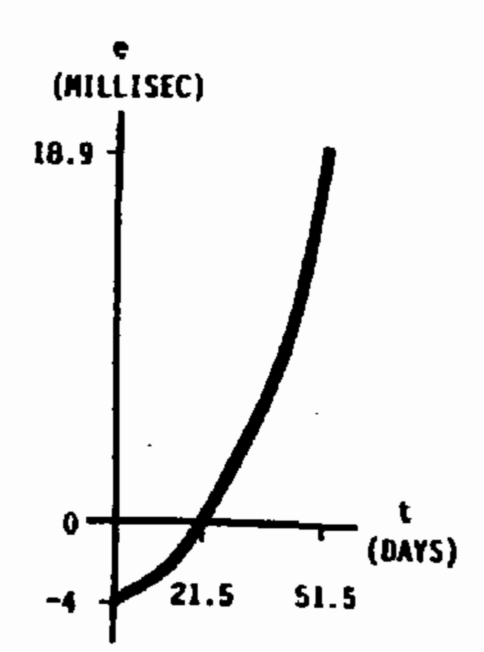
(b)

$$e = \frac{-4 \times 10^{-3}\ \text{SEC}}{86{,}400\ \text{SEC/DAY}} + 0 + \frac{1}{2}\,(2 \times 10^{-10}/\text{DAY})\ (51.5\ \text{DAYS})^2$$

$$= -4.6296 \times 10^{-8}\ \text{DAY} + (10^{-10}/\text{DAY})\ (2652.25\ \text{DAY}^2)$$

$$= 21.8929 \times 10^{-8}\ \text{DAY}$$

$$= 18.9 \times 10^{-3}\ \text{SECOND}$$



10.15

A likelihood function is a maximized conditional probability density function. Under the stated assumptions of zero-mean additive white Gaussian noise (AWGN) and equal energy signals, the conditional probability density of the random variable r with expected value s will be of the form:

$$p(r|s) = \frac{1}{2\pi}\exp\left(\frac{|r-s|^2}{2\pi\sigma}\right)$$

where both r and s are complex, in general. Consider the term $|r-s|^2$. By expressing both r and s in complex form: $r = a + jb$, $s = c + jd$, and expanding and reorganizing, one can derive the equation:

$$|r-s|^2 = |r|^2 + |s|^2 - 2\mathrm{Re}\{rs^*\}$$

where $\mathrm{Re}\{\cdot\}$ is the real part, and the star indicates the complex conjugate. Because the signals are equal energy, the first two terms on the right-hand side will be constant. Thus, only the third term will have a role in the maximization of the probability density. Therefore, the probability density will be maximized when the term: $\Lambda(r|s) = \exp\left(\mathrm{Re}\{rs^*\}\right)$ is maximized, which is in the form of Equation (10.67).

10.16 (a)

The parameters for MSK are $h = ½$, $L = 1$, $M = 2$ and Equation (10.62). The Phase State, Φ_k is defined in Equation (10.61) as:

$$\Phi_k = \pi h \sum_{i=0}^{k-L} \alpha_i \mod 2\pi$$

For MSK, $M = 2$ implies that $\{\alpha_i\}=\{\pm 1\}$, which implies that Φ_k can only take values that are multiples of $\pi/2$ and less than 2π. Thus, $\{\Phi_k\}=\{0,\ \pi/2,\ \pi,\ 3\pi/2\}$.

(b) The modulation phase response for MSK is:

$$q(t)=\begin{cases} 0 & t\le 0 \\ t/2T & 0<t<T \\ 1/2 & t\le T \end{cases}$$

Using this in Equation (10.65) with the other MSK parameters yields:

$$\eta_\ell(t)=\begin{cases} 0 & t\le 0 \\ 2\pi(-1)^\ell (t/2T) & 0<t<T \\ 1/2 & t\le T \end{cases}$$

where $\{\ell\}=\{1,\ 2\}$. Using this result in Equation (10.64) yields:

$$h^{(\ell)}(t)=\begin{cases} \exp(j(-1)^\ell \pi t/T & 0<t<T \\ 0 & \text{elsewhere} \end{cases}$$

which is the desired form of the filters.

(c) Substituting the filter expression from part (b) into Equation (10.66) will yield a form for the parameter $Z_k^{(\ell)}$. Because of the assumed training sequence of alternating plus and minus ones, the results of the summation in Equation (10.66) will repeat after every adjacent pair of incoming symbols. Therefore, without loss of generality, assume that $k = 0, 1$, the first two adjacent symbols in the signal stream. Since the transmitted sequence is known to be alternating, the appropriate filter sequence, which will match the

training sequence, is known to be $\ell = 1,\ 2$. Utilizing these parameters, after a considerable amount of manipulation, terms in the summation of Equation (10.68) can be shown to be of the form:

$$2(T-\tau)\sin\delta\cos(\pi\tau/T),$$

where δ is the phase error: $\delta = \theta - \hat{\theta}$, and τ is the timing error. Similarly, the terms in the summation of Equation (10.69) are of the form:

$$[2\pi(T-\tau)/T]\cos\delta\sin(\pi\tau/T).$$

Thus, for small δ and τ/T, which would be indicative of lock or near lock, the error terms are approximately linear in the error parameters, as one might wish.

10.17

There are clearly no unique answers to this kind of problem, but one approach might be to view the summation on the left-hand side of Equation (10.68) as a function $f(x)$ of the phase error, $\delta = (\theta - \hat{\theta})$. If one then considers the Taylor Series expansion of this function:

$$f(\hat{\theta}+\delta) = f(\hat{\theta}) + \delta f'(\hat{\theta}) + \ldots.$$

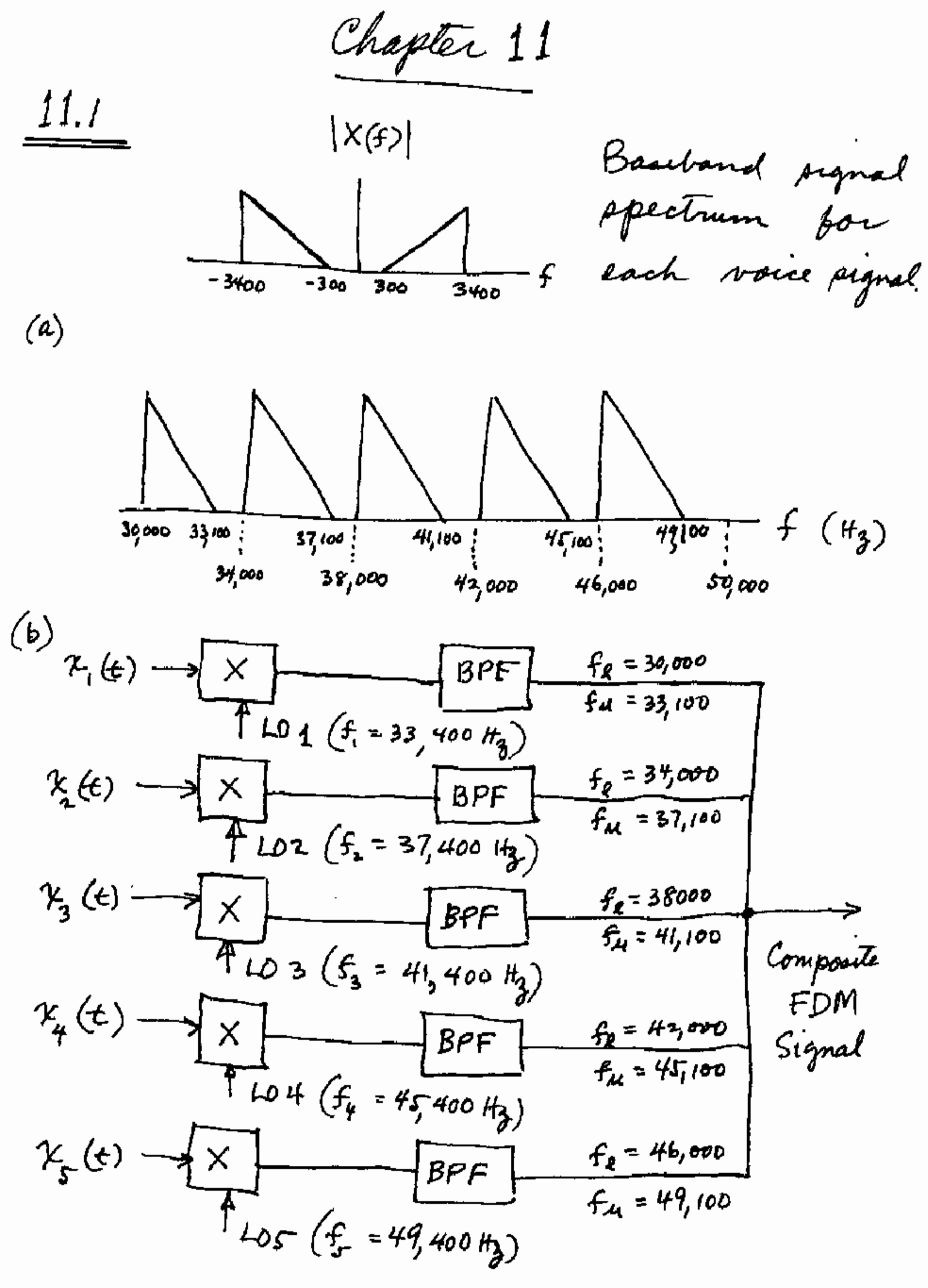
and presumes that δ is small, only the first two terms on the right-hand side need be considered. "Solving" for δ yields:

$$\delta = [f(\hat{\theta}+\delta) - f(\hat{\theta})]/f'(\hat{\theta}).$$

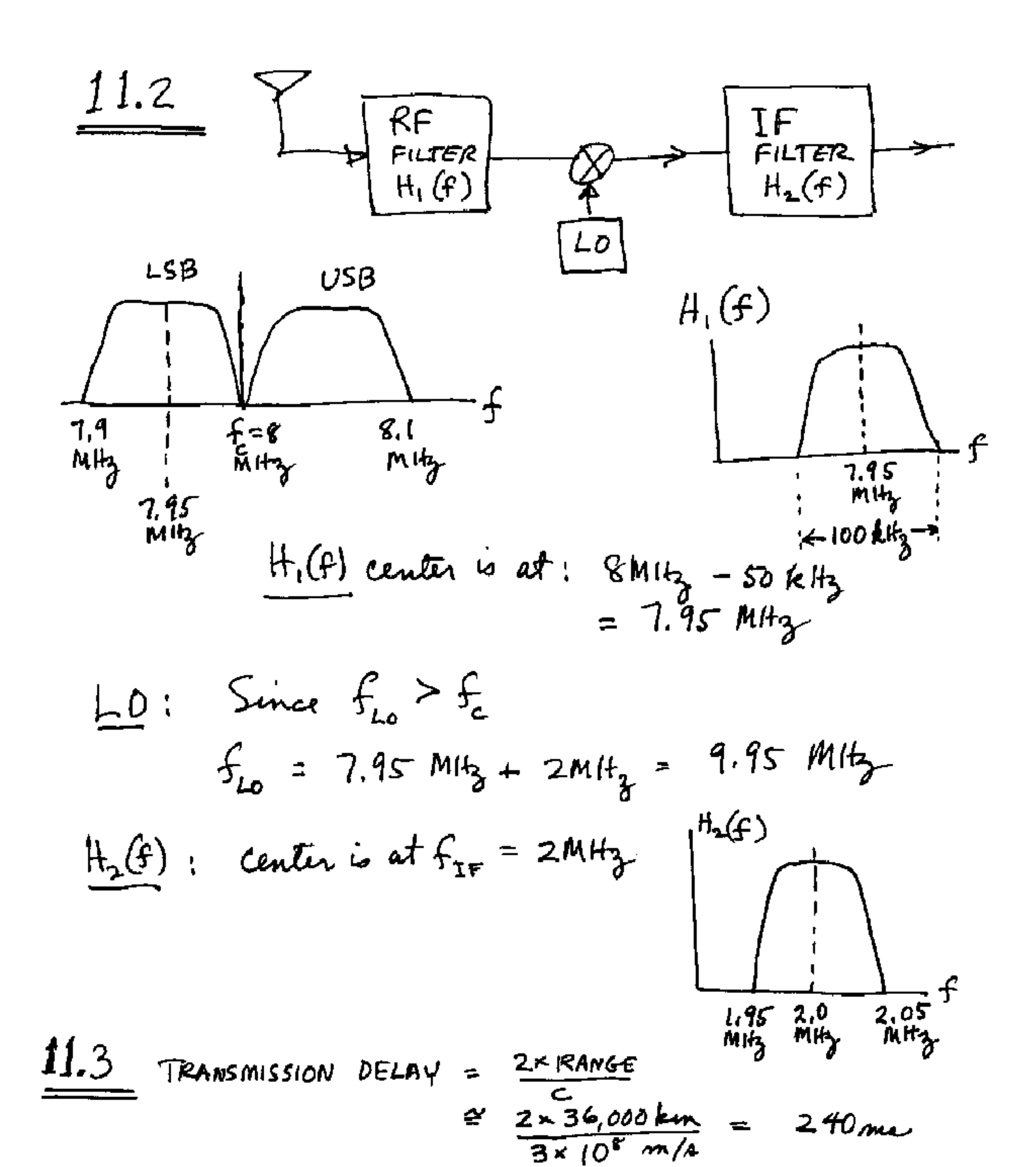
Considering now the idea of successive approximations to the final solution, if $f(\hat{\theta}+\delta)$ is identified with the $(k+1)^{st}$ symbol, and $f(\hat{\theta})$ identified with the k^{th} symbol, the error term for phase could be viewed as being linear in the most recent term in the summation, as suggested by Mengali's iterative approach.

Chapter 11

11.1



11.2



$H_1(f)$ center is at: $8\,\text{MHz} - 50\,\text{kHz} = 7.95\,\text{MHz}$

LO: Since $f_{LO} > f_c$

$$f_{LO} = 7.95\,\text{MHz} + 2\,\text{MHz} = 9.95\,\text{MHz}$$

$H_2(f)$: center is at $f_{IF} = 2\,\text{MHz}$

11.3

TRANSMISSION DELAY $= \dfrac{2 \times \text{RANGE}}{c} \cong \dfrac{2 \times 36{,}000\,\text{km}}{3 \times 10^8\,\text{m/s}} = 240\,\text{ms}$

Comparing Equations (11.13) and (11.14), TDMA delay time savings is $\frac{T}{2}\left(1 - \frac{1}{M}\right) < \frac{T}{2}$. Therefore, the TDMA savings is negligible for frame times of a few msec, not much for frame times of a few tens of msec, and significant only for frame times greater than approx. 100 msec.

11.4 With pure ALOHA, the maximum usable capacity is 0.184×56 kbits/s $= 10.3$ kbits/s (since the delay increases without bound for $\rho \geq 0.184$).

Each station sends 3000 bits/10 sec, or 300 bits/s. Thus, the maximum number of stations that can share this channel is:

$$\left\lfloor \frac{10{,}300 \text{ bits/s}}{300 \text{ bits/s}} \right\rfloor = 34 \text{ stations.}$$

11.5 $\lambda = \dfrac{R \text{ bits/s}}{b \text{ bits/packet}}$; $\lambda_1 = 75$ packets/s, $\lambda_2 = 100$ packets/s, $\lambda_3 = 200$ packets/s

$\lambda_T = \lambda_1 + \lambda_2 + \lambda_3 = 375$ packets/s

$\tau = \dfrac{100 \text{ bits/packet}}{56{,}000 \text{ bits/sec}} = 1.79$ ms/packet

normalized total traffic: $G = \lambda_t \tau = 0.67$

normalized throughput: $\rho = G e^{-2G} = 0.175$

probability of successful transmission:

$P_s = P(K=0) = e^{-2G} = 0.26$

arrival rate of successful packets:

$\lambda = P_s \lambda_t = 0.26 \times 375 \cong 98$ packets/s.

11.6 From Equation (11.28), $\rho = Ge^{-2G}$

$$\frac{d\rho}{dG} = e^{-2G} + \left(-2Ge^{-2G}\right) = e^{-2G}\left(1-2G\right) = 0$$

$\Rightarrow$ extremum is at $G = 0.5$

$$\frac{d^2\rho}{dG^2} = -2e^{-2G} - (1-2G)2e^{-2G}\Big|_{G=0.5} < 0$$

Therefore, the extremum is a maximum

$$\rho_{max} = G_{max}e^{-2G_{max}} = 0.5e^{-1} = \frac{1}{2e}$$

11.7 (a) From Equation (11.24),

$$P(K) = \frac{(\lambda\tau)^K e^{-\lambda\tau}}{K!} \geq 0 \quad \text{for } K \geq 0.$$

Thus, we need only show that $\sum_{K=0}^{\infty} P(K) = 1$

$$\sum_{K=0}^{\infty} P(K) = e^{-\lambda\tau} \sum_{K=0}^{\infty} \frac{(\lambda\tau)^K}{K!} = 1, \text{ since}$$

$\sum_{K=0}^{\infty} \frac{(\lambda\tau)^K}{K!}$ is the Taylor series expansion of $e^{\lambda\tau}$

(b) $$E\{K\} = \sum_{K=1}^{\infty} K\,P(K) = \sum_{K=1}^{\infty} K \frac{(\lambda\tau)^K e^{-\lambda\tau}}{K!}$$

$$= e^{-\lambda\tau} \sum_{K=1}^{\infty} \frac{(\lambda\tau)^K}{(K-1)!} = \lambda\tau e^{-\lambda\tau} \sum_{K=1}^{\infty} \frac{(\lambda\tau)^{K-1}}{(K-1)!} = \lambda\tau$$

(c) Since $P(K)$ is defined as the probability of having exactly K new packets arrive over a time interval of τ seconds, $E\{K\}$ is the average number of new packets that arrives in τ seconds. The average arrival rate is $E\{K\}/\tau = \lambda$, as claimed.

11.8 (a) $$P(N_{n+1}) = \frac{(\lambda_t \tau)^{N_{n+1}} e^{-\lambda_t \tau}}{N_{n+1}!}$$

$$P(N_n) = \frac{(\lambda_t \tau)^{N_n} e^{-\lambda_t \tau}}{N_n!}$$

Joint pdf $= P(N_{n+1})P(N_n) = \dfrac{(\lambda_t \tau)^{N_{n+1}+N_n} e^{-\lambda_t 2\tau}}{N_{n+1}!\ N_n!}$

(b) $P_s = \text{Prob}\,(N_{n+1} = N_n = 0) = \dfrac{(\lambda_t \tau)^{0+0} e^{-\lambda_t 2\tau}}{0!\ 0!} = e^{-\lambda_t 2\tau}$

11.9 $\text{Prob}\,(N = n) = \sum_{m=0}^{n} P(N_n = m)\, P(N_{n+1} = n-m)$

$$= \sum_{m=0}^{n} \frac{(\lambda_t \tau)^m e^{-\lambda_t \tau}}{m!} \frac{(\lambda_t \tau)^{n-m} e^{-\lambda_t \tau}}{(n-m)!}$$

$$= (\lambda_t \tau)^n e^{-\lambda_t 2\tau} \sum_{m=0}^{n} \frac{1}{m!\,(n-m)!}$$

$$= (\lambda_t \tau)^n e^{-\lambda_t 2\tau} \frac{1}{n!} \sum_{m=0}^{n} \binom{n}{m}$$

But, $\sum_{m=0}^{n} \binom{n}{m} = 2^n$ as can be seen by expanding $(a+b)^n$ in its binomial series and letting $a = b = 1$ as follows:

$$(a+b)^n = \sum_{m=0}^{n} a^{n-m} b^m \binom{n}{m}$$

For $a = b = 1 \Rightarrow 2^n = \sum_{m=0}^{n} \binom{n}{m}$

Therefore, $P_N(n) = \dfrac{(\lambda_t \tau)^n e^{-\lambda_t 2\tau} 2^n}{n!}$

$$= \frac{(\lambda_t 2\tau)^n e^{-\lambda_t 2\tau}}{n!}$$

N is the number of packet arrivals over a time interval of length 2τ.

<u>11.10</u> Each station makes an average of 30 requests/hour = 30 requests/3600 s. Therefore, on the average all 6000 stations make

$$\frac{6000 \times 30 \text{ requests}}{3600 \text{ s}} = 50 \text{ requests/s}.$$

Since each slot is 500 μsec in duration, the channel capacity is $1/500\,\mu s$ = 2000 slots/s, and the normalized total traffic, G, is:

$$G = \frac{50 \text{ slot requests/s}}{2000 \text{ slots/s}} = \boxed{0.025}$$

(only a small fraction of the channel capacity).

11.11 (a) The packets that would have arrived in $(T_{n-1}, T_n]$ under pure ALOHA would all arrive at T_n. Similarly, the packets that would have arrived in $(T_n, T_{n+1}]$ under pure ALOHA would arrive at T_{n+1}, as shown below:

The pdfs of N_n and N_{n+1} would not change. (We assume that the underlying packet generating process is Poisson and is independent of the switching algorithm. Slotted ALOHA users simply wait for the next time slot after the packet has been generated.

(b) $$P_s = P(N_n = 0) = \frac{(\lambda_t \tau)^0 e^{-\lambda_t \tau}}{0!}$$

$$= e^{-\lambda_t \tau}$$

Compare this with Equation (11.25).

11.12 (a) $G = \lambda_t \tau$

$= 120$ slot requests/s $\times 12.5$ ms/slot

$= 1.5$

(b) $P_s = P(K=0) = \dfrac{(\lambda_t \tau)^0 e^{-\lambda_t \tau}}{0!} = e^{-G}$

$= e^{-1.5} = 0.22$

(c) Prob of 2 collisions before a successful Transmission $= \left[1 - P(K=0)\right]^2 P(K=0)$

$= \left(1 - e^{-G}\right)^2 e^{-G} = \left(1 - e^{-1.5}\right)^2 e^{-1.5}$

$= 0.135$

11.13 (a) $P_s = P(K=0) = e^{-G}$

$G = -\ln\left[P(K=0)\right]$

With 20% of the slots idle, $P_s = 0.2$

$G = -\ln(0.2) = 1.61$

(b) $\rho = G e^{-G} = 1.61\, e^{-1.61}$

$= 0.32$

(c) $G > 1$

Therefore, the Channel is overloaded.

11.14 Let $P_1(K_1) = \frac{\lambda_1^{K_1} e^{-\lambda_1}}{K_1!}$

$P_2(K_2) = \frac{\lambda_2^{K_2} e^{-\lambda_2}}{K_2!}$

Since τ is arbitrary, we let $\tau = 1$ for notational convenience

$$P_t(K_t) = \sum_{m=0}^{K_t} P_1(m) P_2(K_t - m)$$

$$= \sum_{m=0}^{K_t} \frac{\lambda_1^m e^{-\lambda_1}}{m!} \frac{\lambda_2^{K_t - m} e^{-\lambda_2}}{(K_t - m)!}$$

$$= e^{-(\lambda_1+\lambda_2)} \sum_{m=0}^{K_t} \frac{\lambda_1^m \lambda_2^{K_t-m}}{m!\,(K_t-m)!}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{K_t!} \sum_{m=0}^{K_t} \lambda_1^m \lambda_2^{K_t-m} \binom{K_t}{m}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{K_t!} (\lambda_1 + \lambda_2)^{K_t}$$

Q.E.D.

By induction, the sum of n Poisson processes, with rates $\lambda_1, \lambda_2, \ldots, \lambda_n$ is Poisson with rate $\sum_{i=1}^{n} \lambda_i$.

<u>11.15</u> (a) $W_{max} = \frac{10 \times 10^6 \text{ Hz}}{200 \text{ carriers}} = 50 \text{ kHz/carrier}$

(b) Under the given power-limited condition, the transponder provides a total power, P_t, to the stations, as follows:

$$P_t = 100x + 100(2x) = 300x \text{ Watts}$$

since the smaller stations require twice (3dB) the power of the larger stations. If only large stations are to be serviced, the total 300x watts can provide each of 300 large terminals with x watts each. However, since the total bandwidth of the transponder is 10 MHz, then from the bandwidth consideration, the maximum number of stations are: $N_{max} = \frac{10 \text{ MHz}}{40 \text{ kHz}} = 250$ stations and the transponder is bandwidth limited.

(c) If the 300x watts are to service small stations only, where each station needs 2x watts of power, then $N_{max} = \frac{300x}{2x} = 150$ stations and the transponder is power limited.

11.16 (a) $n = 1$, thus there is no guard time, so that the efficiency = 1.00

For $n > 1$: $Eff = 1 - \frac{n \times 1\,\mu s}{2\,ms}$

Thus, for n = 2, 5, 10, 20, 50, and 100, efficiency is 0.999, 0.9975, 0.995, 0.990, 0.975, and 0.950 respectively

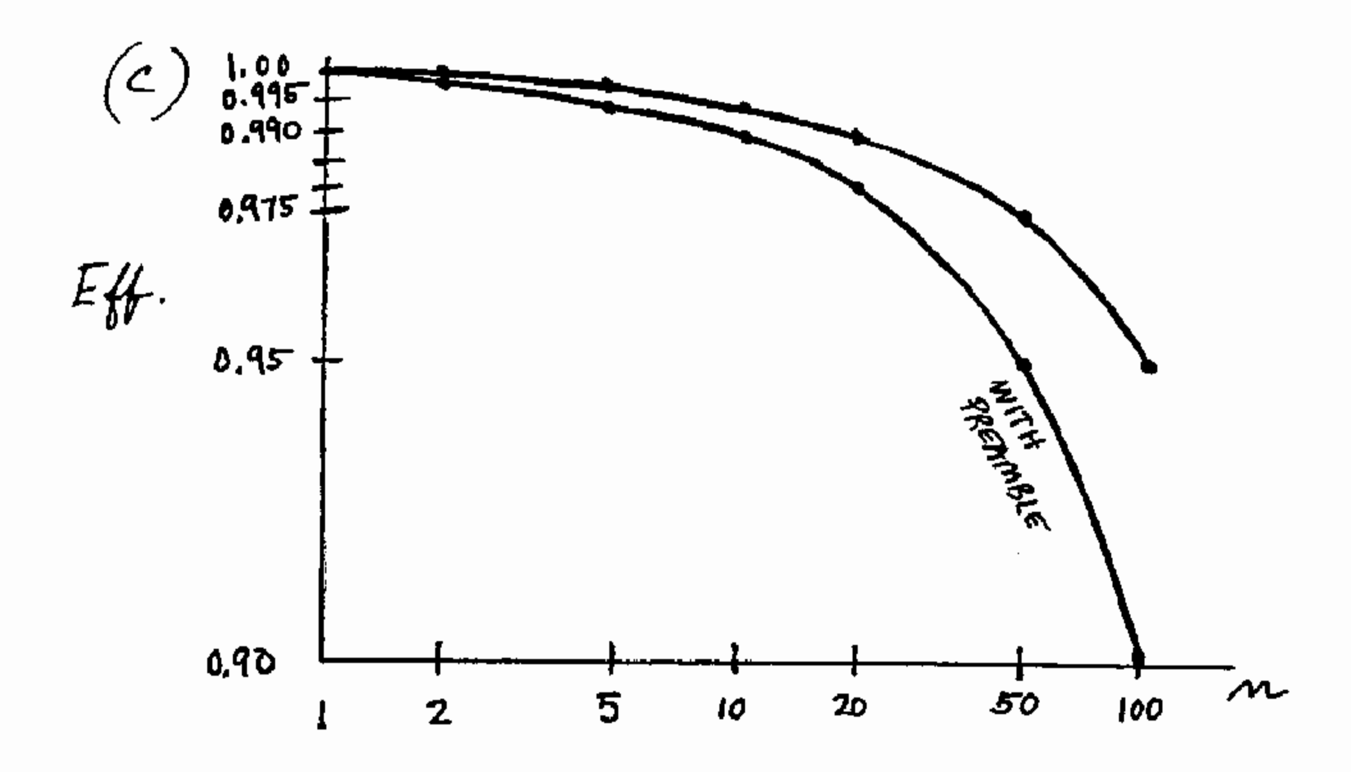
(b) Preamble time = $\frac{10^2 \text{ bits}}{10^8 \text{ bits/sec}} = 10^{-6}$ sec

For $n = 1$, efficiency = $1 - \frac{1\,\mu sec}{2\,msec} = 0.9995$

For $n > 1$: $Eff = 1 - \frac{n \times 2\,\mu s}{2\,ms}$

Thus for n = 2, 5, 10, 20, 50, and 100, efficiency is 0.998, 0.995, 0.990, 0.980, 0.950, and 0.90, respectively.

(c)



<u>11.17</u> (a) Efficiency is maximized because T_{min} is minimized.

(b) A few large S_i or R_j drive efficiency down by increasing T_{min}. They still require frame time when all other users are through. Solution: Divide a large S_i into S_{i1}, S_{i2}, ..., S_{iN} equal pieces, where N is chosen to make S_{ij} about equal in size to the majority of other S_is.

(c) Similar when the preponderance of usage is full duplex (e.g., voice). Dissimilar when the preponderance of usage is simplex (e.g., a data collection net, where many nodes report to a central node).

11.18 (a) At 10 Mbits/s, a bit duration time is 100 ns. In 100 ns, the signal travels a distance of

200 m/μs × 100 ns = **20 meters**

The insertion of one-bit delay by adding a new station is equivalent to an insertion of 20 meters of cable length on the ring.

(b) 10-bit token: If at least 3 stations are on at all times, allowing for 3-bit times of delay, then the ring cable must provide a delay equivalent to the difference of 10-bit times minus 3-bit times, or 7-bit times of delay. Therefore,

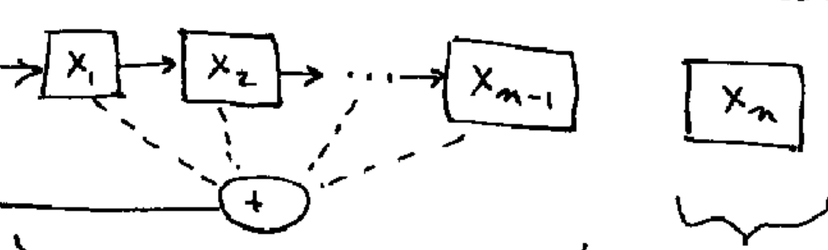
Minimum cable length = 7 × 20 meters (from part a)

= 140 meters

Chapter 12

12.1 n stages implies 2^n n-tuples as possible contents of the register. One n-tuple is the all-zeros state which causes all feedback to be zero, so that the shift register would remain forever in the all-zeros state. Thus, $2^n - 1$ other states exist; in cycling through them, the shift register outputs a maximal length sequence, then must repeat, for there are no other states.

12.2 If the last stage is not an input to the modulo-2 adder, the feedback state will not depend on the last stage. Thus, the last stage will merely act as a delay, and can be conceptualized as separated from the rest of the circuit as illustrated below:

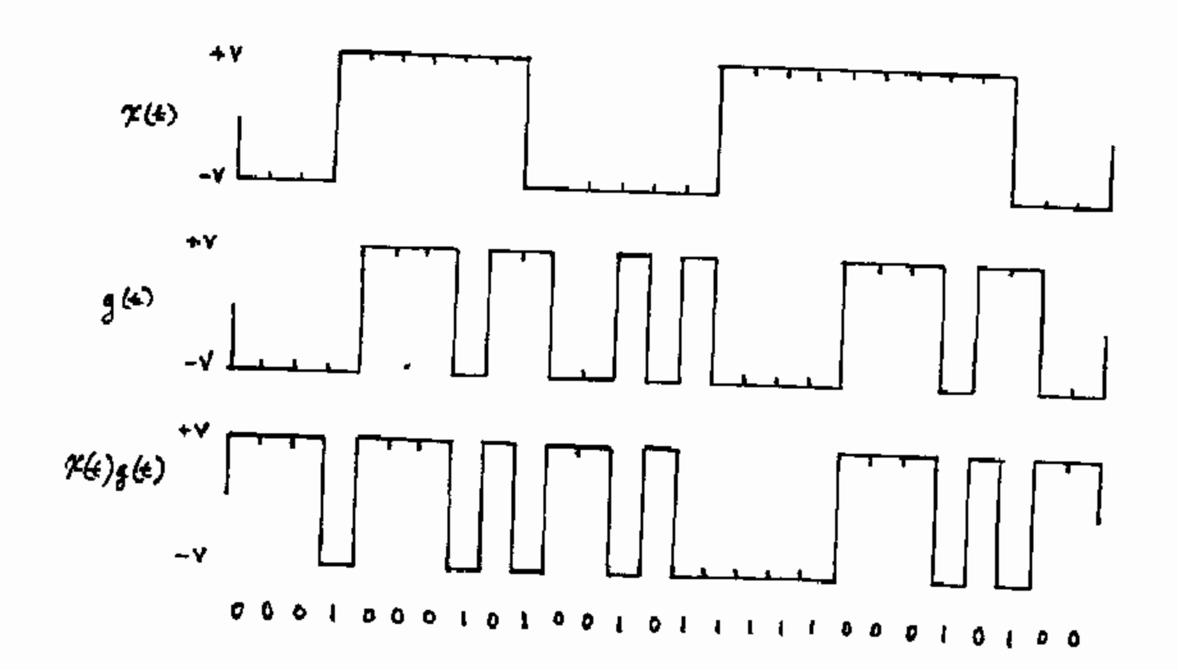


shift register with n-1 stages. last stage acts as a delay.

The resulting circuit has at most $2^{n-1} - 1$ states, and therefore cannot be maximal.

12.3 (a) $x(t) = 100110001$ and
$g(t) = 111100010011010$

Let $+V$ = binary 0, and $-V$ = binary 1



(b) Bandwidth of $x(t)\,g(t) \cong 225$ Hz

(c) Processing gain is $R_p/R = 3$

(d)

$x(t)\,g(t)$: 000100010100101111100010100

$g(t)$ advanced by 1 chip: 011110001001101011110001001

Modulo-2 Sum: 011 010 011 101 000 100 010 011 101

$\hat{x}(t)$ by majority rule: 1 0 [1] 1 [0] 0 0 [1] 1

(e) Majority rule logic is used. A one chip advance in offset produces the above sequence $\hat{x}(t)$, which has 3 bits in error as marked.

12.4 $P_B = 10^{-3} = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) = Q(x)$

From Table B.1, $x = 3.09$

$$\sqrt{\frac{2E_b}{N_0}} = 3.09\ ; \quad E_b/N_0 = 4.77 = 6.8 \text{ dB}$$

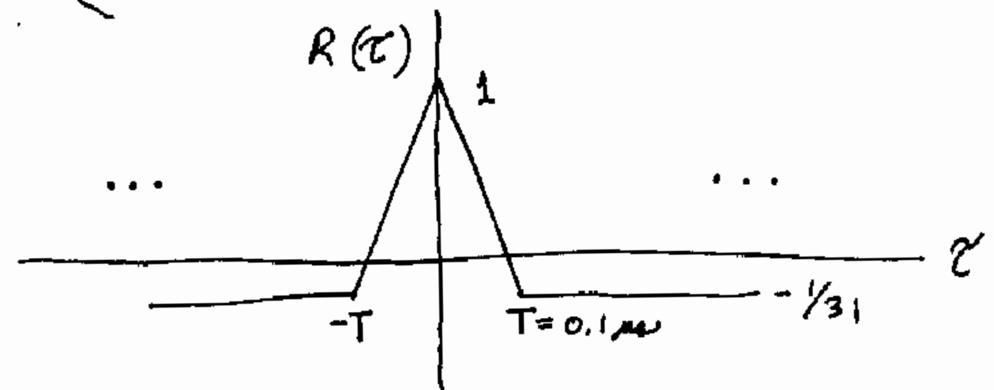
$$\frac{E_b}{J_0} = \frac{G_p}{(J/S)_r} = \frac{R_p/R}{23/1} = 4.77$$

$$R_p = 23 \times 4.77 \times 9.6 \text{ kbits/s} = 1.05 \text{ Mbits/s}$$

12.5 The 31-bit sequence has an autocorrelation function with a maximum value at $\tau = 0$ decreasing linearly to $-1/31$ at $\tau = |T|$ for T equal to a chip interval (0.1 μs in this case).

$$R(\tau) = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)x(t+\tau)\, d\tau$$

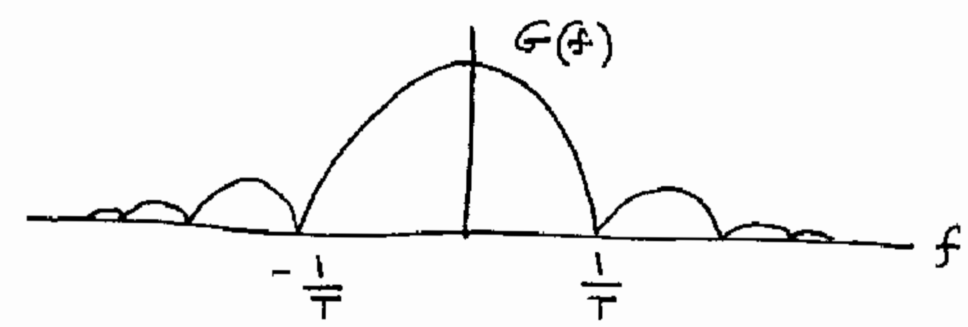
$$= \frac{1}{p}\left\{\begin{array}{l}\text{\# agreements} - \text{\# disagreements in one} \\ \text{full period of the sequence with a } \tau \\ \text{position cyclic shift}\end{array}\right\}$$



$R(\tau)$ repeats for offsets modulo-31 chip times ($T_0 = 3.1$ μs).

$$R(\tau) = \begin{cases} 1 - |\tau|/T & \text{for } |\tau| \le T \\ 0 & \text{otherwise} \end{cases}$$

$$G(f) = \mathcal{F}\{R(\tau)\} = T \,\text{sinc}^2 fT$$



12.6 (a) Hopping bandwidth:

$W_{ss} = (2^{20}-1)$ states $\times 200$ Hz $= 2.1 \times 10^8$ Hz.

(b) Chip rate = hop rate = 2000 chips/s

(c) Chips/symbol: $R_s = \frac{R}{k} = \frac{1200}{3} = 400$

$$\frac{2000 \text{ chips/s}}{400 \text{ symbols/s}} = 5 \text{ chips/symbol}$$

(d) Processing gain: $G_p = W_{SS}/R$

$$= \frac{2.1 \times 10^8 \text{ Hz}}{1200 \text{ bits/s}} = 175{,}000 = 52.4 \text{ dB}$$

12.7 The demodulator for an SFH system is essentially the same as the ones shown in Figures 4.18 and 4.19 with the addition of a frequency

dehopping function at an early stage.

FREQ HOP → ⊗ → LPF → ENV. DET. #1, ENV. DET. #2, ⋮, ENV. DET. #M → DECISION STAGE

There are no accumulators in the block diagram since a detection decision is made after each chip is received. Since there are several symbols per hop, the dehopping proceeds at a slower rate than the symbol detection.

12.8 (a)

The probability of a chip error, using BPSK modulation and a received chip-energy-to-noise-power-spectral-density of 9.6 dB is $P_B = 10^{-5}$ (see Example 6.2). Thus the probability that a chip is correctly received is $(1 - P_B)$ and the probability that a sequence of 100 chips is received correctly is the probability of detection, P_D.

where $P_D = (1 - P_B)^{100} = (1 - 10^{-5})^{100} = 0.999$

The chip duration $T_c = \frac{1}{R_p} = 10^{-7}\,s$, and the number of chips making up the time uncertainty window is

$$N_c = \frac{10^{-3}\,s}{10^{-7}\,s} = 10^4 \text{ chips}$$

Using Equation (10.32) for $\overline{T}_{acq}$ with $P_{FA} = 0$, we get

$$\overline{T}_{acq} = \frac{(2 - P_D)}{P_D} N_c \lambda T_c = \frac{(2 - 0.999)}{0.999} 10^4 \times 10^2 \times 10^{-7}$$

$$= \boxed{100 \text{ ms.}}$$

$$\sigma_{acq}^2 = (2 N_c \lambda T_c)^2 \left(\frac{1}{12} + \frac{1}{P_D^2} - \frac{1}{P_D} \right)$$

$$= (2 \times 10000 \times 100 \times 10^{-7})^2 \left(\frac{1}{12} + \frac{1}{(0.999)^2} - \frac{1}{0.999} \right)$$

$$= 3,37 \times 10^{-3} \text{ sec}^2$$

$$\sigma_{acq} = \sqrt{3,37 \times 10^{-3}}$$

$$= \boxed{58 \text{ ms.}}$$

12.9 (a) $G_p = \frac{R_p}{R} = \frac{100 \text{ kbits/s}}{1 \text{ kbit/s}}$

$G_p = 100 = 20$ dB of processing gain.

Since there are 11 total users, each user will experience interference from ten others, or 10 times the interference provided by any single other user.

Using Equation (12.63)

$$\left(\frac{E_b}{I_0}\right)_r = \frac{G_p}{M-1} = \frac{100}{10} = 10 \text{ (or 10 dB)}$$

(b) If all users double their output power, then $E_b' = 2E_b$ and $I_0' = 2I_0$. Therefore the ratio E_b'/I_0' remains the same. Thus, the ratio E_b/I_0 is independent of the amount of transmit power from the equal-power terminals.

(c) Increasing the users to 101 in number will increase the interferers by an order of magnitude. Therefore, the spread bandwidth (code rate) must also increase by an order of magnitude in order to maintain the original E_b/I_o ratio. $G_p \to 1000$.

<u>12.10</u> (a) $E_b/N_o = 16\,dB = 39.8$ with only one signal transmitted.

To maintain an $\frac{E_b}{N_o + I_o} = 10\,dB = 10$

where I_o is the interference density of other users; we solve for E_b/I_o as follows:

$$\frac{E_b}{N_o + I_o} = \frac{1}{\frac{1}{E_b/N_o} + \frac{1}{E_b/I_o}} = 10$$

$$\frac{1}{E_b/N_o} + \frac{1}{E_b/I_o} = 0.1 \;; \quad \frac{1}{39.8} + \frac{1}{E_b/I_o} = 0.1$$

$$E_b/I_o \cong 13.3 \cong 11.25\,dB$$

$$G_p = \frac{W_{ss}}{R} = \frac{10\,MHz}{10\,kHz} = 10^3 = 30\,dB$$

From Equation (12.41): $\frac{E_b}{I_o} = \frac{G_p}{I/S}$

$$\left(\frac{E_b}{I_0}\right)_{dB} = \left(\frac{S}{I}\right)_{dB} + \left(G_p\right)_{dB}$$

$$11.25 \text{ dB} = \left(S/I\right)_{dB} + 30 \text{ dB}$$

$$\frac{S}{I} = -18.75 \text{ dB} = \frac{1}{75}$$

Therefore 76 total equal-power users can share the band.

(b) With $\frac{E_b}{N_0} = 13 \text{dB}$, $\frac{E_b}{I_0} = 13 \text{dB}$ in order that $\frac{E_b}{N_0 + I_0} = 10 \text{dB}$

$$\frac{E_b}{I_0} = 13 \text{dB} = \left(\frac{S}{I}\right)_{dB} + 30 \text{ dB}$$

$$\frac{S}{I} = -17 \text{dB} = \frac{1}{50}$$

Therefore 51 total equal-power users can now share the band. We see that a reduction in user transmitter power and hence E_b/N_0 means that E_b/I_0 must increase compared to part (a) to fulfill the requirement that $\frac{E_b}{N_0 + I_0} = 10 \text{dB}$. This can only be accomplished by reducing the number of interferers

(or other users). Compare this result with the answer to Problem 12.9 (b) and note that in that problem, negligible receiver noise was assumed, so that the received $E_b/N_0 = \infty$.

(c) If $\frac{E_b}{N_0} \gg 10\,dB$, so that

$$\frac{E_b}{N_0 + I_0} \cong \frac{E_b}{I_0} = 10\,dB$$

$$\frac{E_b}{I_0} = 10\,dB = \left(\frac{S}{I}\right)_{dB} + \left(G_p\right)_{dB}$$

$$\frac{S}{I} = 10\,dB - 30\,dB = -20\,dB$$

$$= 1/100$$

There 101 total equal-power users are the maximum number, assuming N_0 is neglected (E_b/N_0 is very large).

12.11 $\tau = \dfrac{d}{c} = \dfrac{100\ \text{m}}{3 \times 10^8\ \text{m/s}} = 0.33\ \mu\text{s}$

$\therefore$ Minimum chip rate $R_p = \dfrac{1}{\tau} = 3$ megachips/s

12.12 $G_p = \dfrac{R_p}{R} = \dfrac{10\ \text{Mbits/s}}{1\ \text{kbit/s}} = 10^4$

$$E_b/J_0 = \frac{G_p}{J/S} = \frac{G_p}{EIRP_J / EIRP_T}$$

{free space losses are the same for both}

$$E_b/J_0 = \frac{G_p\, EIRP_T}{EIRP_J} = \frac{G_p\, P_T\, A_{eT}}{P_J\, A_{eJ}}$$

$$P_T = \frac{E_b}{J_0}\, \frac{P_J\, A_{eJ}}{G_p\, A_{eT}} = 39.81 \times \frac{4 \times 10^5\, (150)^2}{10^4\, (60)^2}$$

$P_T \cong 10{,}000$ watts

12.13 (a) 75 bits/s $\Longrightarrow$ 150 coded bits/s $\Longrightarrow$ 50 symbols/sec. Therefore, this is a fast hopping system, and the chip rate equals the hopping rate equals 2000 chips/s.

(b) order of diversity $N = \dfrac{2000\ \text{chips/s}}{50\ \text{symbols/s}}$

$= 40$ chips/symbol

(c) With two TDM'd signals

symbol rate = 50 symbols/s × 2 users
= 100 symbols/s

$$N = \frac{2000 \text{ chips/s}}{100 \text{ symbols/s}} = 20 \text{ chips/symb.}$$

Chip rate is the same, the symbol rate is 2× faster and the order of diversity is ½.

(d) With 80 such signals TDM'd

symbol rate = 50 symbols/s × 80 users
= 4000 symbols/s

The hopping rate is still 2000 hops/s. But now the chip duration is dictated by the modulation state changes. There is no diversity since this system can now be classified as a slow hopping system since there are multiple (2) symbols per hop.

12.14 (a) $P_B = \frac{1}{2} \exp\left(-\frac{1}{2} \frac{E_b}{N_0 + J_0}\right)$

$= \frac{1}{2} \exp\left(-\frac{1}{2} \frac{E_b}{101 N_0}\right)$ where $\frac{E_b}{N_0} = 1000$

$P_B = \frac{1}{2} \exp\left(-\frac{1}{2} \times \frac{1000}{101}\right) = \frac{1}{2} e^{-4.95} = 3.5 \times 10^{-3}$

(b) Using Equation (12.50)

$$\rho_0 = \frac{2}{E_b/J_0} = \frac{2(J_0/N_0)}{E_b/N_0} = \frac{200}{1000} = 1/5$$

$$\text{Bandwidth} = 1/5 \times 2\,\text{GHz} = 400\ \text{MHz}.$$

(c) Using Equation (12.51)

$$P_B = \frac{e^{-1}}{E_b/J_0} = \frac{e^{-1}(J_0/N_0)}{E_b/N_0} = \frac{1}{10}e^{-1}$$

$$= 3.68 \times 10^{-2}$$

(d) $P_B = \frac{1}{2}\exp\left(-\frac{1}{2}\,1000\right) = \frac{1}{2}e^{-500} \approx 0$

<u>12.15</u> System temp = 290 K ; $N_0 = kT^{\circ}$

$$N_0 = 1.38 \times 10^{-23} \times 290 = 4 \times 10^{-21}\ \text{W/Hz}$$

$$\rho = \frac{50\ \text{kHz}}{1\ \text{MHz}} = \frac{1}{20}$$

$$E_b = \frac{S}{R} = \frac{10^{-12}\ \text{W}}{3\ \text{bits/symb} \times 3000\ \text{symb/s}} = 1.11 \times 10^{-16}\ \text{Joule}$$

Note that the hopping rate (given as 12,000 hops/s) is not need to solve this problem.

$$J_0' = J/W = \frac{10^{-11}\ \text{W}}{50\ \text{kHz}} = 2 \times 10^{-16}\ \text{W/Hz}$$

Using Equation (12.48)

$$P_B = \frac{(1-\rho)}{2} \exp\left(-\frac{1}{2}\frac{E_b}{N_o}\right) + \frac{\rho}{2}\exp\left(-\frac{1}{2}\frac{E_b}{N_o+J_o}\right)$$

Since E_b/N_o is over 44 dB we conclude that the first part of the right hand side contributes zero probability of error.

$$P_B = \frac{\rho}{2}\exp\left(-\frac{1}{2}\frac{E_b}{N_o+J_o}\right) = \frac{1}{40}\exp\left(-\frac{1}{2}\frac{1.11\times10^{-16}}{N_o+J_o'}\right)$$

We can neglect $N_o = 4\times10^{-21}$ because it is so much smaller than J_o'

$$P_B = \frac{1}{40}\exp\left(-\frac{1}{2}\frac{1.11\times10^{-16}}{2\times10^{-16}}\right) = 1.89\times10^{-2}$$

12.16 (a) For $P_B = 10^{-5}$, $E_b/J_o = 9.6\text{ dB} = 9.12$ (see Example 6.2 in text)

$$\frac{E_b}{J_o} = \frac{W_{ss}/R}{J/S} = \frac{W_{ss}/R}{EIRP_J/EIRP_T}$$

since the propagation losses are the same for both.

$$9.12 = \frac{W_{ss}/10{,}000}{60\text{ kW}/20\text{ kW}} = \frac{W_{ss}}{30{,}000}$$

$$W_{ss} = 274\text{ kHz}$$

(b) $\rho_0 = \frac{0.709}{E_b/J_0} = \frac{0.709}{9.12} = 0.0777$

$(P_B)_{max} = \frac{0.083}{9.12} = 9.1 \times 10^{-3}$

12.17 (a) The time required for a completed hop to get from the terminal to the satellite is $T + h/c$, where T is the duration of a hop,

$$T = \frac{1}{10^4} = 100\,\mu s,$$ and

c is the speed of light. Thus the communicator will be unconditionally safe if: $T + h/c \leq r/c + d/c$

and, the radius of vulnerability is:

$$r = h + Tc - d$$

$$r = h + Tc - \sqrt{h^2 + r^2}.$$ The precise solution involves solving the above quadratic equation.

However, for this problem the parameters are such that $r \ll h$ and $h \cong d$, that we can say:

$$r \cong Tc = 10^{-4}s \times 3 \times 10^{8} \text{ m/s}$$
$$= 30 \text{ km}$$

(b) The precise relationship would be:

$$T + h/c \leq r/c + \tau + d/c$$

where $\tau = 10\ \mu s$ processing time. Thus,

$$r = h + (T - \tau)c - d = h + (T - \tau)c - \sqrt{h^2 + r^2}$$

As in part (a), the parameters are such that $h \cong d$, and we can compute

$$r = (T - \tau)c = (10^{-4} - 10^{-5}) \times 3 \times 10^{8}$$
$$= 27 \text{ km}$$

<u>12.18</u> $T_{hop} \leq \dfrac{d_2 + d_3 - d_1}{c}$; $R_{hop} = \dfrac{1}{T_{hop}}$

where c is the speed of light $\approx 3 \times 10^{8}$ m/s

For an airborne communicator and land based jammer:

$$T_{hop} \leq \frac{d_1 + d_2 - d_3}{c}$$

12.19 $$\frac{P_t}{4\pi d^2} = \frac{100 \text{ W}}{4\pi\,(3.6\times 10^7 \text{ m})^2}$$

$$= 6.14 \times 10^{-15} = -142.1 \text{ dBW/m}^2$$

We can use bandwidth spreading to reduce the -142.1 dBW/m² in a 4 kHz bandwidth to -151 dBW/m².

The bandwidth expansion required is:

$$-142.1 - (-151) = 8.9 \text{ dB-Hz} = 7.76$$

Thus, the spread spectrum bandwidth, W_{ss}, required is:

$$W_{ss} = 7.76 \times 4 \text{ kHz} = 31 \text{ kHz}.$$

12.20 (a) For BFSK (noncoherent),

$$P_B = \frac{1}{2}\exp\left(-\frac{E_b}{2J_0}\right) \quad \text{since } J_0 \gg N_0$$

$$J_0 = J/W_{ss}\ ; \quad E_b = S/R$$

$$P_B = \frac{1}{2}\exp\left(-\frac{S/R}{2J/W_{ss}}\right) = \frac{1}{2}\exp\left(-\frac{S\,G_p}{2J}\right)$$

where $G_p = W_{ss}/R$ is the processing gain or bandwidth expansion.

$$10^{-4} = \frac{1}{2}\exp\left(-\frac{10^{-5}G_P}{2}\right)$$

$$G_p = -\frac{\ln(2\times 10^{-4})}{0.5\times 10^{-5}} = 1.7\times 10^6 = 62.3 \text{ dB}$$

(b) $$P_B = \frac{1}{2}\left\{\frac{1}{2}\exp\left[-\frac{SG_p}{2J(1-\alpha)}\right]\right\} + \frac{1}{2}\left\{\frac{1}{2}\exp\left[-\frac{SG_p}{2J(1+\alpha)}\right]\right\}$$

$$= \frac{1}{4}\left\{\exp\left[-\frac{SG_p}{2J(1-\alpha)}\right] + \exp\left[-\frac{SG_p}{2J(1+\alpha)}\right]\right\}$$

(c)

$$\frac{dP_B}{d\alpha} = \frac{1}{4}\left[-\frac{SG_p}{2J(1-\alpha)^2}\right]\exp\left[-\frac{SG_p}{2J(1-\alpha)}\right] + \frac{1}{4}\left[\frac{SG_p}{2J(1+\alpha)^2}\right]\exp\left[-\frac{SG_p}{2J(1+\alpha)}\right]$$

$$\frac{d^2P_B}{d\alpha^2} = \frac{1}{4}\left[\left(\frac{-SG_p}{2J(1-\alpha)^2}\right)^2 - \frac{SG_p}{J(1-\alpha)^3}\right]\exp\left[-\frac{SG_p}{2J(1-\alpha)}\right]$$
$$+ \frac{1}{4}\left[\left(\frac{-SG_p}{2J(1+\alpha)^2}\right)^2 - \frac{SG_p}{J(1+\alpha)^3}\right]\exp\left[\frac{-SG_p}{2J(1+\alpha)}\right]$$

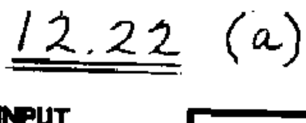
By inspection, $dP_B/d\alpha = 0$ at $\alpha = 0$. Therefore, $\alpha = 0$ is an extremal point in the range $0 \le \alpha \le 1$. From the expression for $d^2P_B/d\alpha^2$ it can be seen that for SG_p/J sufficiently large, $d^2P_B/d\alpha^2 > 0$, $\alpha = 0$ represents a minimum, and $\alpha = 1$ represents a maximum in the range $(0,1)$. Thus, the optimum jammer strategy for large SG_p/J is to jam half the band with the maximum available

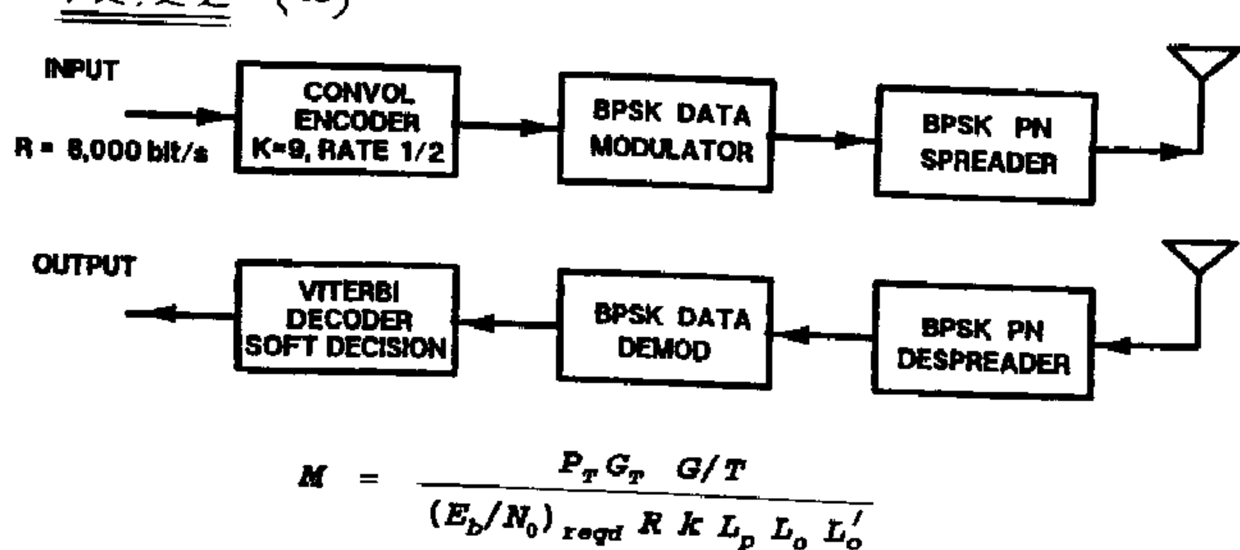
power, and leave the other half of the band completely unjammed.

In the case where SG_P/J is sufficiently small, $d^2P_B/d\alpha^2 < 0$ for all α, $0 \le \alpha \le 1$. In this case, $\alpha = 0$ represents a maximum, and $\alpha = 1$ represents a minimum in the range $(0, 1)$. Thus, the optimum jammer strategy for low SG_P/J is to jam the entire bandwidth equally.

12.21

The primary beneficial attribute of spread-spectrum (SS) systems is interference rejection. For example, such systems were originally developed for rejecting the intentional jamming of an adversary in a military environment. Such an application, where the interferer has a fixed finite amount of interfering power is the typical scenario where spreading the communicator's signal in frequency provides processing gain. But, for the case of AWGN, processing gain is not possible because the power associated with white noise is infinite. That is, however large the SS bandwidth is increased, the noise-power spectral density has the same intensity.





$$M = \frac{P_T G_T \; G/T}{(E_b/N_0)_{reqd} \; R \; k \; L_p \; L_o \; L_o'}$$

WE NOW SOLVE FOR P_T IN DECIBELS:

$$P_T = M + (E_b/N_0)_{reqd} + R + k + L_p + L_o + L_o' - (G_T + G/T)$$

$$= 0 + 4 + 39 - 228.6 + 138.6 + 4 + 30 - (5 - 18) = 0 \text{ dBW}$$

$$= 1 \text{ WATT}$$

(b) WHEN $L_o' = 0$ dB, P_T CAN BE REDUCED TO -30 dBW

= 1 MILLIWATT

(c) $\frac{P_R}{N_0} = \frac{E_b}{N_0} R = \frac{E_{ch}}{N_0} R_{ch}$. THEREFORE, WE CAN COMPUTE:

$$\left(\frac{E_{ch}}{N_0}\right)_{reqd} (dB) = \left(\frac{E_b}{N_0}\right)_{reqd} (dB) + 10 \times \log_{10}\left(\frac{8000}{25 \times 10^6}\right)$$

(d)

$$= 4 \text{ dB} - 35 \text{ dB} = -31 \text{ dB}$$

PROCESSING GAIN, $G_p = \frac{R_{ch}}{R} = \frac{E_b/N_0}{E_{ch}/N_0} = \frac{25 \times 10^6}{8000} = 3125 = 35$ dB

(e)

$$\frac{E_b}{N_0 + I_0} \approx \frac{E_b}{I_0} = \frac{S/R}{I/W_{ss}} = \frac{W_{ss}/R}{I/S} = \frac{G_p S}{I} = \frac{G_p S}{S(N'-1)} = \frac{G_p}{N'-1}$$

$$N' \approx G_p\,(dB) - E_B/I_0\,(dB) = 35 \text{ dB} - 4 \text{ dB} = 31 \text{ dB} \approx 1258$$

12.23 $\frac{E_b}{N_0}(\text{dB}) = \frac{P_R}{N_0}(\text{dB-Hz}) - R(\text{dB-bit/s})$

$$\frac{E_b}{N_0} = 48\ \text{dB-Hz} - (10 \log_{10} 9600)\ \text{dB-bit/s}$$
$$= 8.2\ \text{dB} \quad (\text{or } 6.61)$$

$$\frac{P_R}{N_0} = \frac{E_b}{N_0}R = \frac{E_c}{N_0}R_c = \frac{E_s}{N_0}R_s = \frac{E_{ch}}{N_0}R_{ch}$$

$$\frac{E_{ch}}{N_0} = \frac{P_R}{N_0}\left(\frac{1}{R_{ch}}\right) = \frac{P_R}{N_0}\left(\frac{1}{G_p R}\right) = \left(\frac{1}{G_p}\right)\frac{E_b}{N_0}$$

Since BPSK is the data modulation then each Transmission symbol corresponds to a single channel bit, and we can write

$$\frac{E_s}{N_0} = \frac{E_c}{N_0} = \left(\frac{k}{n}\right)\frac{E_b}{N_0} = \left(\frac{51}{63}\right) \times 6.61 = 5.35$$

Out of the BPSK demodulator, the symbol-error probability, P_E and the channel-bit error probability, p_c, is computed as:

$$p_c = P_E = Q\left(\sqrt{\frac{2E_c}{N_0}}\right) = Q(3.27) = 5.8 \times 10^{-4}$$

Using this value of p_c in Equation (6.46) for the (63, 51) code, yields the decoded bit-error probability of $P_B = 3.6 \times 10^{-7}$. We can therefore verify that we meet the required P_B for this example. Note that the DS/SS has no effect on the error performance of an AWGN channel, and the value of G_p has no bearing on the value of P_B.

$$\frac{E_{ch}}{N_0}(\text{dB}) = \frac{E_b}{N_0}(\text{dB}) - G_p(\text{dB}) = 8.2\ \text{dB} - (10 \log_{10} 1000)\ \text{dB} = -21.8\ \text{dB}$$

The chosen value of $G_p = 1000$ has enabled the DS/SS system to operate at a value of chip energy well below the thermal noise, with the same error performance as without spreading.

12.24

(a) $M = \dfrac{\gamma\; G_v\;\; G_p}{\left(E_b/I_0\right)\, H_0} = \dfrac{1.5\times 2.5}{4\times 1.5} G_p$ where $G_p = \dfrac{3.68\times 10^6}{14.4\times 10^3} = 255.55$

therefore $M = \dfrac{2.5}{4}\times 255.55 \;\; \approx 160$ users/cell

(b) If E_b/I_0 can be lowered by 1 dB (or the factor 1.259), it directly affects the user population by an increase in the same amount. Thus now $M \approx 201$ users/cell.

12.25

$E_{\text{ch}}/I_0 \leq -30.4$ dB. Assume that $E_b/(N_0 + I_0) \approx E_b/I_0$. Then for QPSK modulation with perfect synchronization and $P_B = 10^{-5}$, the required $E_b/I_0 = 9.6$ dB. Then, from the processing gain

$$G_p = \frac{E_b/I_0}{E_{\text{ch}}/I_0} \qquad G_p(\text{dB}) = E_b/I_0(\text{dB}) - E_{\text{ch}}/I_0(\text{dB}) = 9.6 + 30.4 = 40 \text{ dB}$$

We see that for a direct-sequence spread spectrum system, there must be ≥ 10,000 chips/bit to meet these specifications.

12.26

$$G_p = \frac{E_b/I_0}{E_{\text{ch}}/I_0} \qquad E_{\text{ch}}/I_0(\text{dB}) = E_b/I_0(\text{dB}) - G_p(\text{dB}) = 9.6 - 20 = -10.4 \text{ dB}$$

From Equation (12.69), we write:

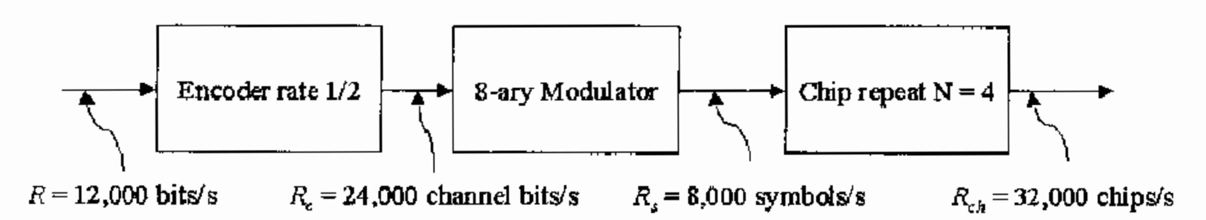
$$\frac{P_r}{I_0} = \frac{E_c}{I_0} R_c = \frac{E_{\text{ch}}}{I_0} R_{\text{ch}} = \frac{E_b}{I_0} R$$

then, $\dfrac{E_c}{I_0} = \left(\dfrac{k}{n}\right)\dfrac{E_b}{I_0}$ since $R_c = \left(\dfrac{n}{k}\right) R$, and

$$\frac{E_c}{I_0}(\text{dB}) = 9.6 \text{ dB} - 3 \text{ dB} = 6.6 \text{ dB}$$

12.27

(a) We start with the chip rate of R_{ch} = 32,000 chips/s and work backwards to find R_s, R_c, and R as follows:



$$R_s = \frac{R_{ch}}{N} = \frac{32{,}000}{4} = 8000 \text{ symbols/s}$$

$$R_c = (\log_2 M)\ R_s = 3\times 8000 = 24{,}000 \text{ channel bits/s}$$

$$R = \frac{k}{n} R_c = \frac{1}{2}\times 24{,}000 = 12{,}000 \text{ bits/s}$$

Given the hopping bandwidth of 1.2 MHz, the processing gain is

$$G_p = \frac{W_{\text{hopping}}}{R} = \frac{1.2\times 10^6}{1.2\times 10^4} = 100 = 20 \text{ dB},$$

and E_b/I_0 is given as 13 dB (= 20). Thus, from Equation (12.69)

$$\frac{P_r}{I_0} = \frac{E_b}{I_0} R = \frac{E_c}{I_0} R_c = \frac{E_s}{N_0} R_s = \frac{E_{ch}}{N_0} R_{ch}$$

Therefore, $\frac{P_r}{I_0} = 20\times 12{,}000 = 240{,}000$ (53.8 dB-Hz), and

$$\frac{E_{ch}}{I_0} = \frac{P_r/I_0}{R_{ch}} = \frac{240{,}000}{32{,}000} = 7.5 \text{ (8.8 dB)}$$

$$\frac{E_s}{I_0} = \frac{240{,}000}{R_s} = \frac{240{,}000}{8000} = 30 \text{ (14.8 dB)}$$

$$\frac{E_c}{I_0} = \frac{240{,}000}{R_c} = \frac{240{,}000}{24{,}000} = 10 \text{ (10 dB)}$$

(b) This system will meet the FCC Part 15 processing gain requirement. However, the hopping bandwidth of 1.2×10^6 exceeds the maximum-bandwidth per channel requirement.

12.28

Using Equation (12.69),

$$\frac{P_r}{I_0} = \frac{E_b}{I_0} R = 4 \times 20{,}000 = 80{,}000 \ \text{(49 dB-Hz)}$$

$$\frac{E_c}{I_0} = \left(\frac{k}{n}\right)\frac{E_b}{I_0} = \frac{1}{2} \times 4 = 2 \ \text{(3 dB)}$$

$$R_c = 2 \times R = 40{,}000 \ \text{channel bits/s}$$

Each 256-ary waveform represents 8 channel bits. Hence,

$$\frac{E_W}{I_0} = 8 \times \frac{E_c}{N_0} = 16 \ \text{(12 dB)}$$

$$R_W = \frac{R_c}{8} = \frac{40{,}000}{8} = 5000 \ \text{Walsh waveforms/s}$$

$$R_{\text{wch}} = 256 \times R_W = 256 \times 5000 = 1.28 \times 10^6 \ \text{Walsh chips/s}$$

$$\frac{E_{\text{wch}}}{I_0} = \frac{E_W}{I_0}\left(\frac{R_W}{R_{\text{wch}}}\right) = \frac{16}{256} = 0.0625 \ (-12 \text{ dB})$$

R_{ch} is given as 10.24 Mchips/s

$$\therefore \ \frac{E_{\text{ch}}}{I_0} = \frac{P_r}{I_0}\left(\frac{1}{R_{\text{ch}}}\right) = \frac{80{,}000}{1.024 \times 10^7} = 0.0078 \ (-21 \text{ dB})$$

The processing gain is: $G_p = \dfrac{R_{\text{ch}}}{R} = \dfrac{1.024 \times 10^7}{2 \times 10^4} = 512$

The ratio of spread spectrum chips to Walsh chips is:

SS chips/Walsh chip $= \dfrac{R_{\text{ch}}}{R_{\text{wch}}} = \dfrac{1.024 \times 10^7}{1.28 \times 10^6} = 8$

Chapter 13

13.1 $H(X) = -\sum_i p_i \log_2 p_i$

$= -(0.9 \log_2 0.9 + 0.08 \log_2 0.08 + 0.02 \log_2 0.02)$

$= 0.9 \times 0.152 + 0.08 \times 3.644 + 0.02 \times 5.644$

$= 0.541$ bit

13.2
(a)

$P(A) = P(A|A)\,P(A) + P(A|B)\,P(B)$

$P(B) = P(B|A)\,P(A) + P(B|B)\,P(B)$

$P(A) + P(B) = 1$; $P(A|A) + P(B|A) = 1$

$P(B|B) + P(A|B) = 1$

$P(A) = [1 - P(B|A)]\,P(A) + P(A|B)\,[1 - P(A)]$

$P(A) = P(A) - P(A)\,P(B|A) + P(A|B) - P(A)\,P(A|B)$

$$P(A)\left[P(A|B) + P(B|A)\right] = P(A|B)$$

$$P(A) = \frac{P(A|B)}{P(A|B) + P(B|A)} = \frac{0.6}{0.6 + 0.2} = 0.75$$

$$P(B) = 1 - P(A) = 0.25$$

(b) $H(X) = P(A)\,H(X|A) + P(B)\,H(X|B)$

$$H(X|A) = -\left[P(A|A)\log_2 P(A|A) + P(B|A)\log_2 P(B|A)\right]$$

$$H(X|B) = -\left[P(B|B)\log_2 P(B|B) + P(A|B)\log_2 P(A|B)\right]$$

$$H(X|A) = -\left[0.8\log_2 0.8 + 0.2\log 0.2\right]$$

$$= 0.722 \text{ bit}$$

$$H(X|B) = -\left[0.4\log_2 0.4 + 0.6\log_2 0.6\right]$$

$$= 0.971 \text{ bit}$$

$$H(X) = 0.75 \times 0.722 + 0.25 \times 0.971$$

$$= 0.784 \text{ bit}$$

(c) $H(X) = -\left[P(A)\log_2 P(A) + P(B)\log_2 P(B)\right]$

$$= -\left[0.75\log_2 0.75 + 0.25\log_2 0.25\right]$$

$$= 0.811 \text{ bit}$$

13.3 (a) 2^{16} levels; ±5 volts equals a 10 volt range. Therefore,

$$q = \frac{10v.}{2^{16}} = 1.526 \times 10^{-4}\ v$$
$$= 152.6\ \mu V$$

(b) $$\sigma_q = \frac{q}{\sqrt{12}} = 44.05\ \mu V$$

(c) SNR (full-scale sinusoid)

$$\sigma_s = \frac{1}{\sqrt{2}} = 0.707$$

$$SNR = \frac{\sigma_s}{\sigma_q} = \frac{0.707}{44.05 \times 10^{-6}} = 16,050.5$$

$$SNR_{dB} = 20 \log_{10} (SNR)_{VOLTAGE} = 84.11\ dB$$

(d) $$\sigma_d = \frac{E_{max}}{2^{16}\sqrt{12}} = \frac{100\ miles \times 5280\ ft/mile}{2^{16}\sqrt{12}}$$
$$= 2.33\ feet$$

13.4 10-Bit Converter; ±5 volts

(a) $$q = \frac{2E_{max}}{2^{10}} = 10 \times 2^{-10} = 9.77\ mV$$

(b) 5 volt sinusoid; rms signal power = $\frac{E_{max}^2}{2}$

$$\text{rms noise power} = \frac{q^2}{12} = \frac{1}{12}\left(\frac{2E_{max}}{2^{10}}\right)^2$$

$$= \frac{1}{3}\left(\frac{E_{max}^2}{2^{20}}\right)$$

$$SNR = \frac{E_{max}^2/2}{\frac{1}{3}\left(\frac{E_{max}^2}{2^{20}}\right)} = \left(\frac{3}{2}\right)2^{20} = \underline{61.97\,dB}$$

(c) $$\text{Signal power} = \frac{\left(\frac{1}{100}E_{max}\right)^2}{2} \Rightarrow \begin{matrix}40\,dB\\ \text{Attenuation}\end{matrix}$$

$$\therefore \text{Output SNR} = 61.97 - 40 = \underline{21.97\,dB}$$

(d) $4\sigma = E_{max}$; $\sigma = \text{rms signal} = \frac{E_{max}}{4}$

$$\text{rms signal power} = \frac{E_{max}^2}{16}$$

$$SNR = \frac{E_{max}^2/16}{\frac{1}{3}\left(\frac{E_{max}^2}{2^{20}}\right)} = \left(\frac{3}{16}\right)2^{20}$$

$$= \underline{52.94\,dB}$$

(e) $$\text{Prob of saturation} = 2\int_4^{\infty}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx$$

$$\cong \frac{2}{\sqrt{2\pi}\,x}e^{-x^2/2}\Big]_{x=4} = \underline{6.7\times10^{-5}}$$

13.5 $C(x) = K\int \sqrt[3]{p(x)}\,dx$ for x in the range $(-4, 4)$. We solve for $C(x)$ in the interval $x=0$ to $x=4$, and construct the overall compression function by symmetry.

$$C(x) = \underbrace{\int K\sqrt[3]{6A}\,dx}_{\text{interval }(0,1)} + \underbrace{\int K\sqrt[3]{4A}\,dx}_{\text{interval }(1,2)} + \underbrace{\int K\sqrt[3]{2A}\,dx}_{\text{interval }(2,3)} + \underbrace{\int K\sqrt[3]{A}\,dx}_{\text{interval }(3,4)}$$

$$C(x) = K(0.61337x + C_1) + K(0.5358x + C_2)$$
$$+ K(0.4253x + C_3) + K(0.3376x + C_4)$$

The boundary conditions of each segment of $C(x)$ must be equal to its neighboring segment boundary conditions. Therefore,

at $x = 0$: $C_1 = 0$

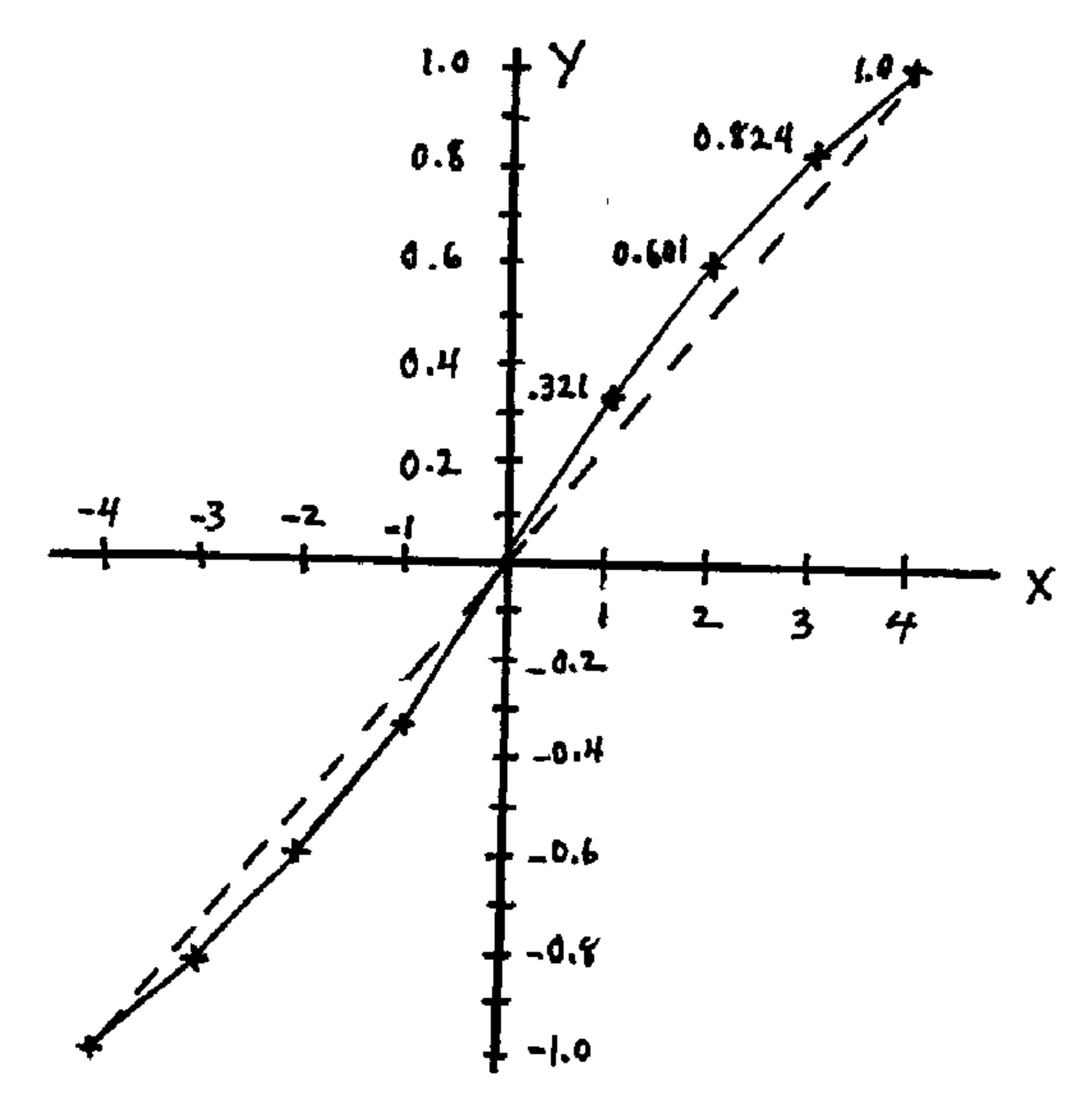
at $x = 1$: $0.61337x = 0.5358x + C_2$
$C_2 = 0.0776$

at $x=2$: $0.5358x + 0.0776 = 0.4253x + C_3$
$C_3 = 0.2986$

at $x=3$: $0.4253x + 0.2986 = 0.3376x + C_4$
$C_4 = 0.5617$

at $x=4$: $K(0.3376x + 0.5617) = 1$
$$K = \frac{1}{1.912}$$

$$C(x) = \underbrace{0.321x}_{\text{interval }(0,1)} + \underbrace{0.280x + 0.041}_{\text{interval }(1,2)}$$
$$+ \underbrace{0.2224x + 0.1562}_{\text{interval }(2,3)} + \underbrace{0.1766x + 0.2438}_{\text{interval }(3,4)}$$



13.6 (a) $SNR = \dfrac{1}{\log_e^2(1+\mu)(q^2/12)}$

if signal > $\dfrac{E_{max}}{\mu} = \dfrac{5 \text{ volts}}{100} = 50$ mV.

$SNR = \dfrac{1}{\log_e^2(1+100)(2^{-20}/12)}$

$= 590,765 = 57.7$ dB

(b) If signal < $\frac{E_{max}}{\mu} = 50\,mV$, then with $\mu = 100$ there is only a small amount of compression. In this region with $\mu = 100$ the device acts more like a linear quantizer.

$$SNR = \frac{\frac{1}{2}\left(\frac{E_{max}}{100}\right)^2}{\frac{1}{12}\left(\frac{2E_{max}}{2^{-10}}\right)^2} = 157.3 = 22\ dB$$

(c) If $\mu = 250$, the SNR will be the same at 5 volts and 50 mV input signals. Thus, for both parts (a) and (b)

$$SNR = \frac{12 \cdot 2^{20}}{\log_e^2(1+250)} = 412,140 = 56,2\ dB$$

13.7 (a) $q = \frac{2E_{max}}{2^{16}} = 2^{-15} E_{max}$

Signal power $= E_{max}^2/2$

Quantization noise power $= (1/12)\, q^2$

$= (\frac{1}{12})\, 2^{-30} E_{max}^2 = (1/3)\, 2^{-32} E_{max}^2$

$$SNR = \frac{E^2_{max}}{2} \cdot 3 \cdot 2^{32} \cdot \frac{1}{E^2_{max}}$$

$$= \frac{3}{2} \cdot 2^{32} = 98.09 \text{ dB}$$

(b) $rms = \frac{peak}{20} = \frac{E_{max}}{20}$

Signal power $= (rms)^2 = \frac{E^2_{max}}{400}$

Quantization noise power $= \frac{1}{3} \cdot 2^{-32} \cdot E^2_{max}$

$\therefore SNR = \frac{E^2_{max}}{400} \cdot 3 \cdot 2^{32} \cdot \frac{1}{E^2_{max}} = \frac{3}{400} \cdot 2^{32}$

$= 75.07$ dB

(c) Input bit rate $= 44.1 \times 10^3$ samples/s $\times$ 2 channels $\times$ 16 bits/sample

$= 1.411 \times 10^6$ bits/s

Include 100% overhead:

Output bit rate $= 2.822 \times 10^6$ bits/s

(d) # bits/hour $= 2.822 \times 10^6$ bits/s $\times$ 3600 s/hour

$= 10.16$ Gigabits/hour

(e) 1500 pages × 2 columns/page × 100 lines/col × 7 words/line × 6 letters/word × 6 bits/letter

$= 7.56 \times 10^7$ bits/dictionary

(approximately 27 seconds of recording)

∴ a disc contains the equivalent of 134 comparable-books storage capacity.

<u>13.8</u> Solution will be given using two methods. Method #1 — in the time domain:

A sin θ
X_0
$-X_0$
θ

$$\epsilon^2 = \frac{1}{\pi}\int_0^{\pi} (A\sin\theta - X_0)^2 \, d\theta$$

$$= \frac{1}{\pi}\int_0^{\pi} (A^2\sin^2\theta - 2AX_0\sin\theta + X_0^2) \, d\theta$$

$$= \frac{1}{\pi}\left\{\left[\left(\frac{A^2}{2} + X_0^2\right)\theta - 2AX_0\cos\theta\right]_0^{\pi}\right\}$$

$$= \frac{A^2}{2} + X_0^2 - \frac{4AX_0}{\pi}$$

$$\frac{d\epsilon^2}{dX_o} = 2X_o - \frac{4A}{\pi} = 0$$

$$\therefore X_o = \frac{2A}{\pi}$$

$$\epsilon^2_{MIN} = \frac{A^2}{2} + \frac{4A^2}{\pi^2} - \frac{4A}{\pi} \cdot \frac{2A}{\pi} = \frac{A^2}{2} - \frac{4A^2}{\pi^2}$$

$$= \frac{A^2}{2}\left(1 - \frac{8}{\pi^2}\right) = \frac{A^2}{2}(0.189) = 0.095\, A^2$$

Method # 2 - Density domain

$$\epsilon^2 = \int_0^A (x - x_o)^2 \frac{1}{\pi\sqrt{A^2 - x^2}}\, dx$$

This is the pdf for a sine wave with amplitude A, and uniform random phase $(0, 2\pi)$.

$$\epsilon^2 = \int_0^A \frac{x_o^2 - 2xx_o + x^2}{\pi\sqrt{A^2 - x^2}}\, dx$$

$$= \frac{x_o^2}{2} - \frac{2x_o A}{\pi} + \frac{A^2}{4}$$

$$\frac{d\epsilon^2}{dx_o} = x_o - \frac{2A}{\pi} = 0$$

$$\therefore x_o = \frac{2A}{\pi} \checkmark$$

13.9

$\alpha = C(1) = \dfrac{R(1)}{R(0)}$

X(n), $\epsilon(n)$, Q(x), $\tilde{\epsilon}(n-1)$, $\hat{X}(n)$, $\tilde{X}(n-1)$, DELAY, α, $\tilde{X}(n)$

$$X(n) = \sin\left(\frac{2\pi}{10} n + \phi\right)$$

$$R(1) = E\{X(n)\, X(n-1)\}$$

$$= E\left\{\sin\left(\frac{2\pi}{10} n + \phi\right) \sin\left(\frac{2\pi}{10}(n-1) + \phi\right)\right\}$$

$$= E\left\{\frac{1}{2}\cos\frac{2\pi}{10} - \frac{1}{2}\cos\left[\frac{2\pi}{10}(2n-1) + 2\phi\right]\right\}$$

$$= \frac{1}{2}\cos\frac{2\pi}{10} = 0,405$$

$$R(0) = \frac{1}{2}\ ; \quad C(1) = \frac{R(1)}{R(0)} = \cos\frac{2\pi}{10} = 0,809$$

Input power: $R(0) = \dfrac{A^2}{2}$

Output power: $R_d(0) = R(0)\left[1 - C^2(1)\right]$

$$= \frac{A^2}{2}\left(1 - 0.809^2\right) = 0.346\,\frac{A^2}{2}$$

$$R_d(0)/R(0) = 0.346 = -4.6\ dB$$

13.10 (a) $\hat{X}(n) = a_1 X(n-1) + a_2 X(n-2)$

$$\epsilon^2(n) = \left[X(n) - a_1 X(n-1) - a_2 X(n-2)\right]^2$$

$$E\{\epsilon^2(n)\} = \sigma_\epsilon^2 = R(0)\left[1 + a_1^2 + a_2^2\right]$$

$$+ R(1)\,2a_1a_2 - R(1)\,2a_1 - R(2)\,2a_2$$

Taking partials with respect to a_1 and a_2, then setting to zero,

$$a_1^{OPT} = \frac{R(0)R(1) - R(1)R(2)}{R^2(0) - R^2(1)}$$

$$a_2^{OPT} = \frac{R(0)R(2) - R(1)R(1)}{R^2(0) - R^2(1)}$$

Since $C(1) = \dfrac{R(1)}{R(0)}$ and $C(2) = \dfrac{R(2)}{R(0)}$

$$a_1^{OPT} = \frac{C(1) - C(1)C(2)}{1 - C^2(1)}$$ and

$$a_2^{OPT} = \frac{C(2) - C^2(1)}{1 - C^2(1)}$$

(b) Substituting a_1^{OPT} and a_2^{OPT} for a_1 and a_2 respectively in σ_ϵ^2,

$$\sigma_\epsilon^2 = \left[(1 + a_1^{OPT^2} + a_2^{OPT^2}) + 2a_1^{OPT}a_2^{OPT}C(1) - 2a_1^{OPT}C(1) - 2a_2^{OPT}C(2)\right]R(0)$$

$$= \left\{1 - C^2(1) - \frac{\left[C^2(1) - C(2)\right]^2}{1 - C^2(1)}\right\}R(0)$$

(c) $C(n) = 1 - \frac{|n|}{4}$, $C(1) = 3/4$, $C(2) = 1/2$;

Substituting into σ_ϵ^2 above, we get

$$\sigma_\epsilon^2 = 0.428\, R(0)$$

(d) $C(n) = \cos(\theta_0 n)$, $C(1) = \cos\theta_0$, $C(2) = \cos 2\theta_0$

$$\sigma_\epsilon^2 = (1 - \cos^2\theta_0) - \frac{\left[\cos^2\theta_0 - \cos 2\theta_0\right]^2}{1 - \cos^2\theta_0}$$

$$= \sin^2\theta_0 - \frac{\left[\frac{1}{2} - \frac{1}{2}\cos 2\theta_0\right]^2}{\sin^2\theta_0}$$

$$= \sin^2\theta_0 - \frac{\left[\sin^2\theta_0\right]^2}{\sin^2\theta_0} = 0$$

Therefore, the prediction is perfect.

13.11 (a)

Improvement in the sigma-delta modulator SNR for the case of a noise transfer-function with a single zero is 9-dB per doubling of sample rate. When operated at 20 times the Nyquist rate, the improvement is:

$$9\text{ dB} \times \log_2(20) = 4.329 \times 9\text{ dB} = 38.9\text{ dB}$$

Output SNR = 6 dB + 38.9 dB = 44.9 dB
(equivalent to a 7-bit conversion)

(b)

Improvement in SNR for the same modulator in part (a) that is operated at 50 times the Nyquist rateo is:

$$9\text{ dB} \times \log_2(50) = 5.644 \times 9\text{ dB} = 50.8\text{ dB}$$

Output SNR = 6 dB + 50.8 dB = 56.8 dB
(equivalent to a 9-bit conversion)

(c) Improvement in SNR for two-zero sigma delta modulator is 15-dB per doubling of sample rate: therefore improvement is

$$15\text{ dB} \times \log_2(20) = 4.329 \times 15\text{ dB} = 64.8\text{ dB}$$

Output SNR = 6 dB + 64.8 dB = 70.8 dB
(equivalent to a 12-bit conversion

13.12

A	0.9	
B	0.08	
C	0.02	

A - 0
B - 10
C - 11

$\bar{n} = (0.9)(1) + (0.08)(2) + (0.02)(2) = 1.1$ bits/symbol

13.13

AA	0.81
AB	0.072
BA	0.072
AC	0.018
CA	0.018
BB	0.0064
BC	0.0016
CB	0.0016
CC	0.0004

AA - 0
AB - 11
BA - 100
AC - 1011
CA - 10100
BB - 101010
BC - 1010111
CB - 10101100
CC - 10101101

	n_i	$n_i P_i$
AA	1	0.81
AB	2	0.144
BA	3	0.216
AC	4	0.072
CA	5	0.090

	n_i	$n_i P_i$
BB	6	0.0384
BC	7	0.0112
CB	8	0.0128
CC	8	0.0032

$$\bar{n} = \sum_i n_i P_i = 1.3976$$

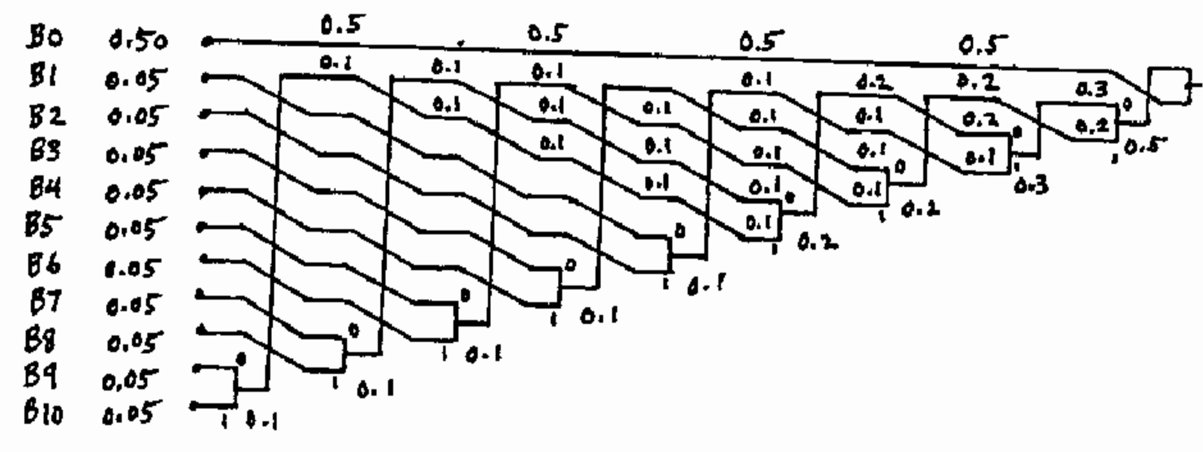
$$\bar{n}_1 = \frac{\bar{n}}{2} = 0.6988 \text{ bits/input symbol}$$

13.14

(a) 100 equally-likely characters

$2^6 < 100 < 2^7$; therefore we require 7-bit codewords

(b) Huffman Code: We shall define B_0 to be the collection of the last 90 characters



	CODE	
B_0 -	1	Prefix for 7-bit code identifying the last 90 characters
B_1 -	0010	$n_i = 4$
B_2 -	0011	4
B_3 -	0100	4
B_4 -	0101	4
B_5 -	0110	4
B_6 -	0111	4
B_7 -	00000	5
B_8 -	00001	5
B_9 -	00010	5
B_{10} -	00011	5

$\bar{n}_i = 4.4$, 50%

$\bar{n}_0 = 8$, 50%

$$\bar{n} = 8 \times 0.5 + 4.4 \times 0.5 = 6.2 \text{ bits/character}$$

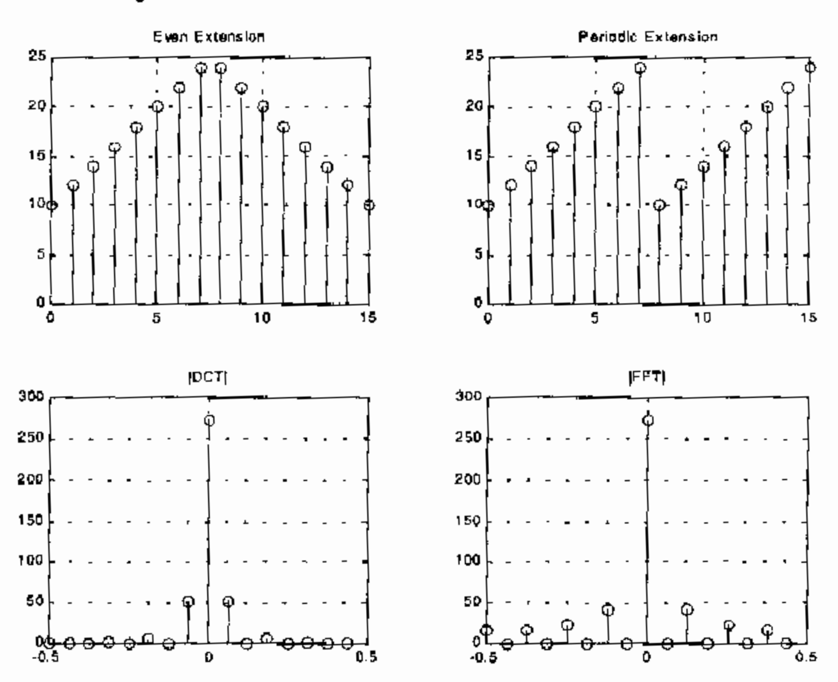
13.15

INPUT	CODE	n
1W	000111	6
1B	010	3
2W	0111	4
2B	11	2
4W	1011	4
4B	011	3
8W	10011	5
8B	000101	6
16W	101010	6
16B	0000010111	10
32W	00011011	8
32B	000001101010	12
{ 64W	11011	5
0W }	00110101	8
{ 64B	0000001111	10
0B }	000110111	9
{ 128W	10010	5
0W }	00110101	8
{ 128B	000011001000	12
0B }	000110111	9
{ 256W	0110111	7
0W }	00110101	8
{ 256B	000001011011	12
0B }	000110111	9
{ 512W	01100101	8
0W }	00110101	8
{ 512B	0000001101100	13
0B }	000110111	9
1W	000111	6
		215 bits

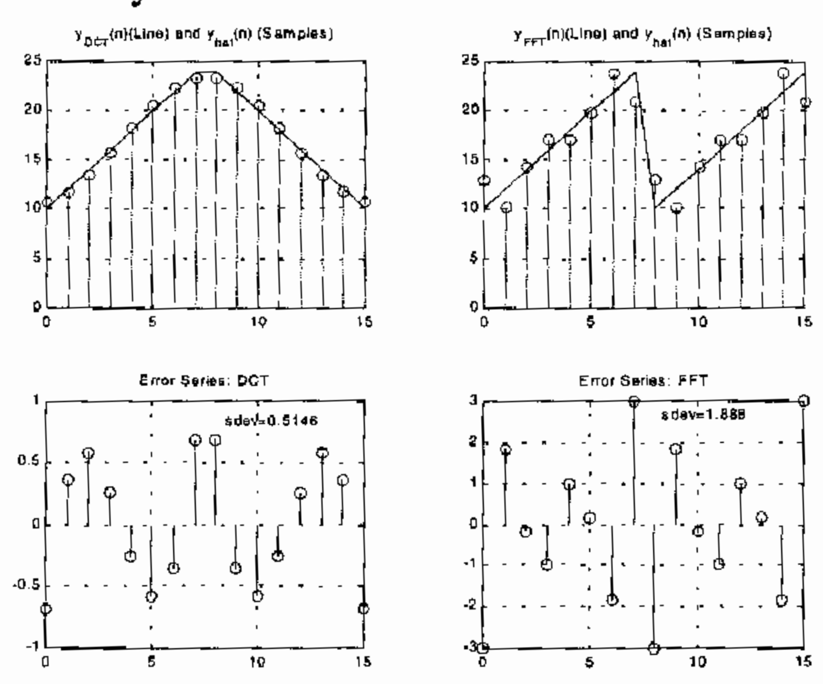
Output = 215 bits
Input = 2047 bits

} Ratio = 0.105 coded bits to input bits

13.16 Time and spectral description of even and periodically extended data sets



Time and error sequences from quantized even and periodically extended data sets.



13.17 S(0,0) = 11001100 , S(1,0) =010101, S(0,1) = 110001, and S(2,1) = 00110. The raster scan would deliver the binary sequence:

11001100	010101	00000	0000	000	00	00	00
110001	00000	0000	000	00	00	0	0
00110	0000	000	00	00	0	0	0
0000	000	00	00	0	0	0	0
000	00	00	0	0	0	0	0
00	00	0	0	0	0	0	0
00	0	0	0	0	0	0	0
00	0	0	0	0	0	0	0

The raster scan with run length of zeros identified by Huffman terminating codewords delivers 72 bits as:

11001100 010101 0100111 (18-zeros) 110001 0100111 (18-zeros) 00110 110100 (14-zeros) 110101 (15-zeros) 001000 (12-zeros) 00111 (10-zeros) 10100 (9-zeros) 10100 (9-zeros)

The zig-zag scan would deliver the following binary sequence:

13.17 (cont'd.)

11001100								
010101	110001							
00110	00000	00000						
0000	0000	0000	0000					
000	000	000	000	000				
00	00	00	00	00	00	00		
00	00	00	00	00	00	00	00	
00	0	0	0	0	0	0	0	00
	0	0	0	0	0	0	0	0
		0	0	0	0	0	0	0
			0	0	0	0	0	0
				0	0	0	0	0
					0	0	0	0
						0	0	0
							0	0
								0

The zigzag scan with run length of zeros identified by Huffman terminating code words delivers 37 bits as follows:

11001100 010101 110001 00110 11011 (64 -zeros)
0100101 (54-zeros)

13.18 The 8x8 block of 8-bit input samples requires 512 bits. The fully populated DCT required 133 bits for a compression ratio of 512/136 or 3.8 or approximately 2.1 bits per pixel. The lightly populated DCT requires 35 populated bits plus 12 bits for run length code, 1101 (64-zeros) and 00010110 (37-zeros), for a total of 37 bits. The compression ratio is 512/37 or 13.8 or approximately 0.6 bits per pixel.

Chapter 14

14.1 $H(X) = \sum_{X} P(X) \log_2 \frac{1}{P(X)}$

$P(0 \le X \le 2^{16}-1) = \frac{1}{2} \times \frac{1}{2^{16}} = 0.5 \times 2^{-16}$

$P(2^{16} \le X \le 2^{32}-1) = \frac{1}{4} \times \frac{1}{2^{16}} = 0.25 \times 2^{-16}$

$P(2^{32} \le X \le 2^{64}-1) = \frac{1}{4} \times \frac{1}{2^{32}} = 0.25 \times 2^{-32}$

where X is an integer random variable.

$H_1(x) = 2^{16} \times 0.5 \times 2^{-16} \log_2 (2 \times 2^{16})$

$= 0.5 \times \log_2 2^{17} = 8.5$

$H_2(x) = 2^{16} \times 0.25 \times 2^{-16} \log_2 (4 \times 2^{16})$

$= 0.25 \log_2 2^{18} = 4.5$

$H_3(X) = 2^{32} \times 0.25 \times 2^{-32} \log_2 (4 \times 2^{32})$

$= 0.25 \times \log_2 2^{34} = 8.5$

$H(X) = H_1(x) + H_2(x) + H_3(x) = 8.5 + 4.5 + 8.5$

$= 21.5$ bits

14.2 (a) $H(X) = \sum_{X} P(X) \log_2 \frac{1}{P(X)}$

$H(X) = 4 \times \frac{1}{4} \times \log_2 4 = 2$ bits

(b) $H(X|Y) = \sum_{Y} P(Y) \sum_{X} P(X|Y) \log_2 \frac{1}{P(X|Y)}$

$= 2\left(\frac{1}{2}\right)\left[(2)\frac{3}{8}\log_2 \frac{8}{3} + (2)\frac{1}{8}\log_2 8\right]$

$= \frac{6}{8} \times 1.42 + \frac{2}{8} \times 3 = 1.82$ bits

14.3 Using Equation (14.5)

$H(M) = -(5 \times 0.116 \log_2 0.116 + 7 \times 0.06 \log_2 0.06)$

$= 3.5$ bits/letter

Redundancy in the language, on the average:

$D = H(M) - r = 3.5 - 1.5 = 2$ bits/letter

where r is the true rate of the language, taken to be 1.5 bits/letter, as in English.

$H(K) = \log_2 (12!) = 28.8$ bits

Unicity distance $N \cong \frac{H(K)}{D}$

$= \frac{28.8 \text{ bits}}{2 \text{ bits/letter}} \cong 15$ letters

14.4 (a) $H(K) = \log_2 (26)^{10}$

$= \log_2 1.41 \times 10^{14} = 47$ bits

Unicity distance $N \cong \frac{H(K)}{D} = \frac{47 \text{ bits}}{3.2 \text{ bits/character}}$

$\cong 15$ characters

where the redundancy in the English language is taken to be $D = 3.2$ bits/char.

(b) $H(K) = \log_2 (26 P 10) = 44.13$ bits

where $(n P_r) = \frac{n!}{(n-r)!}$ means the permutation of n things taken r at a time.

$N \cong \frac{H(K)}{D} = \frac{44.13 \text{ bits}}{3.2 \text{ bits/char.}} \cong 14$ characters

14.5 (a) $H(K) = \log_2 (1000)^{10} = 99.66$ bits

$N \cong \frac{H(K)}{D} = \frac{99.66 \text{ bits}}{3.2 \text{ bits/char}} \cong 31$ characters

(b) Key digits may have no duplicates

$H(K) = \log_2 (1000 P 10) = 99.59$ bits

$N \cong \frac{99.59 \text{ bits}}{3.2 \text{ bits/char}} \cong 31$ characters

14.6 (a) $N \cong \frac{H(K)}{D} = \frac{56 \text{ bits}}{3.2 \text{ bits/char}}$

$= 17.5$ characters

(b) $N \cong \frac{128 \text{ bits}}{3.2 \text{ bits/char}} = 40$ characters

14.7 YES. A contiguous sequence of P-boxes can be replaced by a single P-box. Similarly, a contiguous sequence of S-boxes can be replaced by a single S-box.

14.8 $L_0 = 32$ zeros and $R_0 = 32$ zeros

The R_0 sequence is extended to 48 zeros and then modulo-2 summed with 48 key stream zeros.

The resulting 48 zeros are partitioned into eight groups, B_j ($j = 1, \cdots, 8$) of six

bits each, $B_j = b_1, b_2, b_3, b_4, b_5, b_6$.
Bits $b_1 b_6$ select a row from the S_j region of the S-box, and bits $b_2 b_3 b_4 b_5$ select a column. For each of these eight groups the row-column (0,0) is selected. The S-box output, therefore, yields the sequence:

14, 15, 10, 7, 2, 12, 4, 13

Converted to a binary sequence, we get the following 32 bit sequence

1110 1111 1010 0111 0010 1100 0100 1101

The result is permuted using the P-table (Table 14.4) into:

11011000 11011000 1101 1011 1011 1100.

The above sequence is modulo-2 added to the 32 bit all-zeros L_0 sequence. Thus, the output of the first iteration L_1, R_1, Consists of:

32 left-half zeros followed by the right-half 32 bit sequence

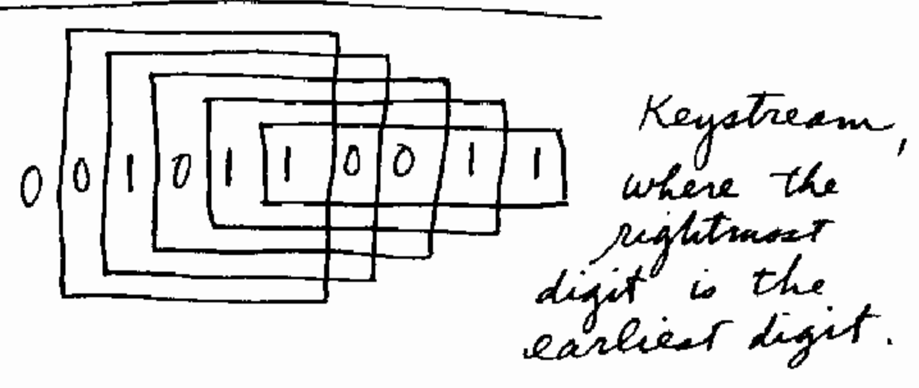
13, 8, 13, 8, 13, 11, 11, 12

(expressed here as 16-ary numbers).

14.9

01 01 1 0 10 01 Plaintext

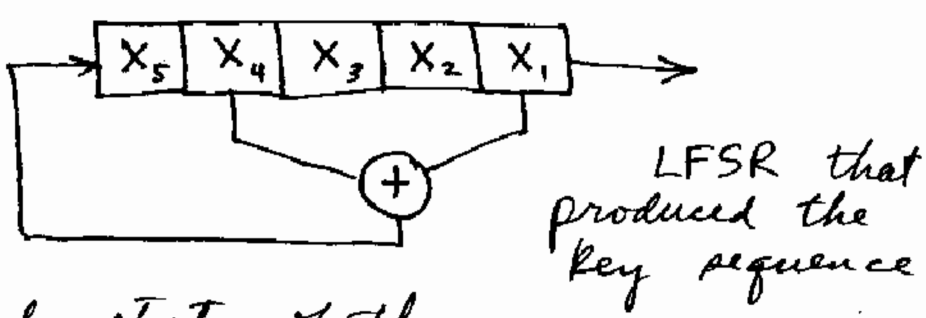
01 11 01 10 10 Ciphertext



Keystream, where the rightmost digit is the earliest digit.

$$g_5 + g_2 + g_1 = 1$$
$$g_5 + g_4 + g_1 = 0$$
$$g_4 + g_3 = 1$$
$$g_5 + g_3 + g_2 = 0$$
$$g_4 + g_2 + g_1 = 0$$

Solving these 5 simultaneous equations, we get: $g_1 = 1$, $g_2 = 0$, $g_3 = 0$, $g_4 = 1$, $g_5 = 0$



LFSR that produced the key sequence

The initial state of the register is: 1 0 0 1 1

where the rightmost bit is the earliest bit.

To determine if the sequence is maximal length, let the LFSR run, starting with the above initial state. Count to verify if $2^5 - 1 = 31$ shifts are required to return it to the initial state.

10011	11010
11001	11101
01100	01110
10110	10111
01011	11011
00101	01101
10010	00110
01001	00011
00100	10001
00010	11000
00001	11100
10000	11110
01000	11111
10100	01111
01010	00111
10101	10011 ← Repeat of the starting state.

Since there are 31 shifts, the LFSR is maximal length.

14.10

i	x_i	a_i	b_i	y_i
0	2668	1	0	
1	151	0	1	17
2	101	1	−17	1
3	50	−1	18	2
4	1	3	−53	

Since b_4 is negative,

$$e = b_4 + 2668 = 2615$$

14.11

$$(M^e \text{ modulo-}n)^d \text{ modulo-}n = M^{ed} \text{ modulo-}n$$

ed modulo-$\varphi(n) = 1$ implies

$ed = k\,\varphi(n) + 1$ for some integer k.

Thus, M^{ed} modulo-n = $M^{k\varphi(n)+1}$ modulo-n

$= M M^{k\varphi(n)}$ modulo-n

$= M \left(M^{k\varphi(n)} \text{ modulo-}n\right)$ modulo-n

where $M^{k\varphi(n)}$ modulo-n = $\left(M^{\varphi(n)} \text{ modulo-}n\right)^k$ mod-n

$M^{\varphi(n)}$ modulo-n $= 1$ is known as Euler's theorem. (See references [3, 4, 13] Chapter 14).

Then, $M^{k\varphi(n)}$ modulo-n $= 1^k$ modulo-n

$= 1$

And $M^{k\varphi(n)+1}$ modulo-n $= M$

Thus, $\left(M^e \text{ modulo-}n\right)^d$ modulo-n $= M$

14.12 $\varphi(n) = (p-1)(q-1) = 4 \times 6 = 24$

ed modulo-24 $= 1$

$11\, e$ modulo-24 $= 1$

i	x_i	a_i	b_i	y_i
0	24	1	0	
1	11	0	1	2
2	2	1	-2	5
3	1	-5	11	

Thus $\boxed{e = 11}$

Check: ed modulo-$\varphi(n) = 1$

11×11 modulo-24 $= 1$ ✓

14.13 (a) 13, 17, 19, 23, 29

(b)

i	x_i	a_i	b_i	y_i
0	360	1	0	
1	37	0	1	
2	27	1	-9	1
3	10	-1	10	2
4	7	3	-29	1
5	3	-4	39	2
6	1	11	-107	

$b_6 = -107$; $e = 360 - 107 = \boxed{253}$

Check: 37×253 modulo-360 $= 1$ ✓

The word DIGITAL must be numerically encoded so that no digit exceeds $n-1 = 402$. We can use a simple code that replaces each letter with a two digit number in the range (01, 26) corresponding to its position in the alphabet: Thus, DIGITAL becomes: 04090709200112. The message needs to be encrypted two digits at a time using $C = (M)^e$ mod-n.

14.14 (a) $a' = 1, 3, 5, 10, 20$

$M = 51$, $W = 37$. Find the inverse of W modulo-51, as follows:

i	x_i	a_i	b_i	y_i
0	51	1	0	
1	37	0	1	1
2	14	1	-1	2
3	9	-2	3	1
4	5	3	-4	1
5	4	-5	7	1
6	1	8	-11	

$$W^{-1} = b_6 + 51 = 51 - 11 = 40$$

Check: 37×40 modulo-51 $= 1$ ✓

$a_i = a_i' W$ modulo-M

$a_1 = 1 \times 37$ modulo-51 $= 37$

$a_2 = 3 \times 37$ modulo-51 $= 9$

$a_3 = 5 \times 37$ modulo-51 $= 32$

$a_4 = 10 \times 37$ modulo-51 $= 13$

$a_5 = 20 \times 37$ modulo-51 $= 26$

$\underline{a} = 37, 9, 32, 13, 26$

$\underline{X} = 1\ 1\ 0\ 1\ 1$

$\underline{S} = 37 + 9 + 13 + 26 = 85$

(b) $\underline{S}' = \underline{a}' \cdot \underline{X}$

$= W^{-1} \underline{S}$ modulo-M

$= 40 \times 85$ modulo-51 $= 34$

Thus the authorized receiver easily transforms $\underline{S}' = 34$, using $\underline{a}' = 1, 3, 5, 10, 20$

into the message: 1 1 0 1 1

14.15

The public key of the recipient is $y = g^a \bmod n = 3^4 \bmod 17 = 13$, and the message is encrypted as follows:

$$
\begin{aligned}
y_1 &= g^k \bmod n = 3^2 \bmod 17 = 9 \\
y_2 &= M \times (y^k \bmod n) = 7 \times (13^2 \bmod 17) \\
&= 7 \times 16 = 112.
\end{aligned}
$$

The ciphertext pair is (9, 112). Decryption of this ciphertext yields the message, as follows: $M = y_2/(y_1^a \bmod n) = 112/(9^4 \bmod 17) = 112/16 = 7$.

14.16

Subkeys: $Z_1^1 = 0003$, $Z_2^1 = 0002$, $Z_3^1 = 0003$, $Z_4^1 = 0002$, $Z_5^1 = 0003$, $Z_6^1 = 0002$, $Z_1^2 = 0003$, and $Z_2^2 = 0002$. $M_1 = 6E6F$. $M_2 = M_3 = M_4 = 0000$

1. $M_1 \times Z_1$ = 6E6F × 0003 = 4B4C (modulo $2^{16} + 1$ multiplication).
2. $M_2 + Z_2$ = 0000 + 0002 = 0002.
3. $M_3 + Z_3$ = 0000 + 0003 = 0003.
4. $M_4 \times Z_4$ = 0000 × 0002 = 0000.
5. Result from steps (1) and (3) are XOR'ed: 4B4C XOR 0003 = 4B4F.
6. Result from steps (2) and (4) are XOR'ed: 0002 XOR 0000 = 0002.
7. Result from step (5) × Z_5: 4B4F × 0x0003 = E1ED.
8. Results from steps (6) and (7) are added: 0002 + E1ED = E1EF.
9. Result from step (8) and Z_6 are multiplied: E1EF × 0002 = C3DD.
10. Results from steps (7) and (9) are added: E1ED + C3DD = A5CA (modulo 2^{16}).

11. Results from steps (1) and (9) are XOR'ed:
    4B4C XOR C3DD = 8891.
12. Results from steps (3) and (9) are XOR'ed:
    0003 XOR C3DD = C3DE.
13. Results from steps (2) and (10) are XOR'ed:
    0002 XOR A5CA = A5C8.
14. Results from steps (4) and (10) are XOR'ed:
    0000 XOR A5CA= A5CA.

Thus, the output of the first round is: 8891 C3DE A5C8 A5CA.

14.17

From Section 14.5.3.1, d = 157, whose binary representation is: 10011101.

We use the Square-and-Multiply technique for C_{11} = 2227, shown in the table below. We thus decrypt as in Section 14.5.3.1, which yields the plaintext $M_{11} = (2227)^{157}$ modulo-2773 = 32

Row Number	Binary of d (MSB first)	Modulo multiplication (modulo 2773)
0		1
1	1	$1^2 \times 2227 = 2227$
2	0	$2227^2 = 1405$
3	0	$1405^2 = 2422$
4	1	$2422^2 \times 2227 = 2461$
5	1	$2461^2 \times 2227 = 267$
6	1	$267^2 \times 2227 = 807$
7	0	$807^2 = 2367$
8	1	$2367^2 \times 2227 = 32$

15.1

(a) The distribution function is found by integrating the pdf over the desired region. Thus we can write

$$F_R(r_0) = P(R \le r_0) = \int_0^{r_0} p(u)\, d(u) = \int_0^{r_0} \frac{u}{\sigma^2} \exp\left[-\frac{u^2}{2\sigma^2}\right] du$$

We use the properties of integrals associated with the exponential function to give us:

$$F_R(r_0) = \left[-\exp\left(-\frac{u^2}{2\sigma^2}\right)\right]_0^{r_0} = 1 - \exp\left(-\frac{r_0^{\,2}}{2\sigma^2}\right)$$

(b) The rms value is given by $\sqrt{2}\sigma$. Now for the case where the signal level is 15 dB below the rms value:

$$10\log\left(\frac{r_0}{\sqrt{2}\sigma}\right) = -15\,\text{dB} \qquad \left(\frac{r_0}{\sqrt{2}\sigma}\right) = 3.16\times 10^{-2}$$

Now from part a):

$$F_R(r_0) = 1 - \exp\left(-\left(\frac{r_0}{\sqrt{2}\sigma}\right)^2\right) = 1 - \exp\left(-(0.0316227)^2\right)$$

The percent of time that the signal level is 15 dB below the rms value is equal to 0.09995%

(c) The rms value is given by $\sqrt{2}\sigma$. Now for the case where the signal level is 5 dB below the rms value:

$$10\log\left(\frac{r_0}{\sqrt{2}\sigma}\right) = -5\,\text{dB} \qquad \left(\frac{r_0}{\sqrt{2}\sigma}\right) = 3.16\times 10^{-1}$$

Now from part (a):

$$F_R(r_0) = 1 - \exp\left(-\left(\frac{r_0}{\sqrt{2}\sigma}\right)^2\right) = 1 - \exp\left(-(0.316227)^2\right)$$

The percent of time that the signal level is 5 dB below the rms value is equal to 9.52%

15.2

(a) First, we need to calculate the rms delay spread which is given by:

$$\sigma_\tau = \sqrt{\overline{\tau^2} - (\bar{\tau})^2} = \sqrt{1.8\times10^{-10} - 10^{-10}} \;=\; 8.94\ \mu s$$

Coherence bandwidth for a correlation of at least 0.9 is given by:

$$f_0 \approx \frac{1}{50\sigma_\tau} = \frac{1}{50\times8.94\times10^{-6}} = 2.24\ \text{kHz}$$

(b)

$$f_0 \approx \frac{1}{5\sigma_\tau} = \frac{1}{5\times8.94\times10^{-6}} = 22.37\ \text{kHz}$$

(c) Assume the bandwidth is equal to the symbol rate = 20kHz. Using the results of b) for the dense scatterer model (50% correlation) yields that $f_0 > W$. However, the values of f_0 and W are so close, that we best call this case marginally frequency selective.

15.3

(a) The mean excess delay can be calculated as follows

$$\bar{\tau} = \frac{\sum_k P(\tau_k)\tau_k}{\sum_k P(\tau_k)} = \frac{\sum_k a_k^2\tau_k}{\sum_k a_k^2} = \frac{(1)(2)+(0.1)(3)}{(1+0.1+0.01)} = \frac{2.3}{1.11} = 2.072\,\mu s$$

(b) The second moment is given by:

$$\overline{\tau^2} = \frac{\sum_k P(\tau_k)\tau_k^2}{\sum_k P(\tau_k)} = \frac{\sum_k a_k^2\tau_k^2}{\sum_k a_k^2} = \frac{(1)(2)^2+(0.1)(3)^2}{(1+0.1+0.01)} = \frac{4.9}{1.11} = 4.14\,\mu s^2$$

(c) The rms delay spread is given by the following:

$$\sigma_\tau = \sqrt{\overline{\tau^2} - (\bar{\tau})^2} = \sqrt{4.14 - (2.072)^2} = 0.35\,\mu s$$

(d) The 90% coherence bandwidth is given by the following:

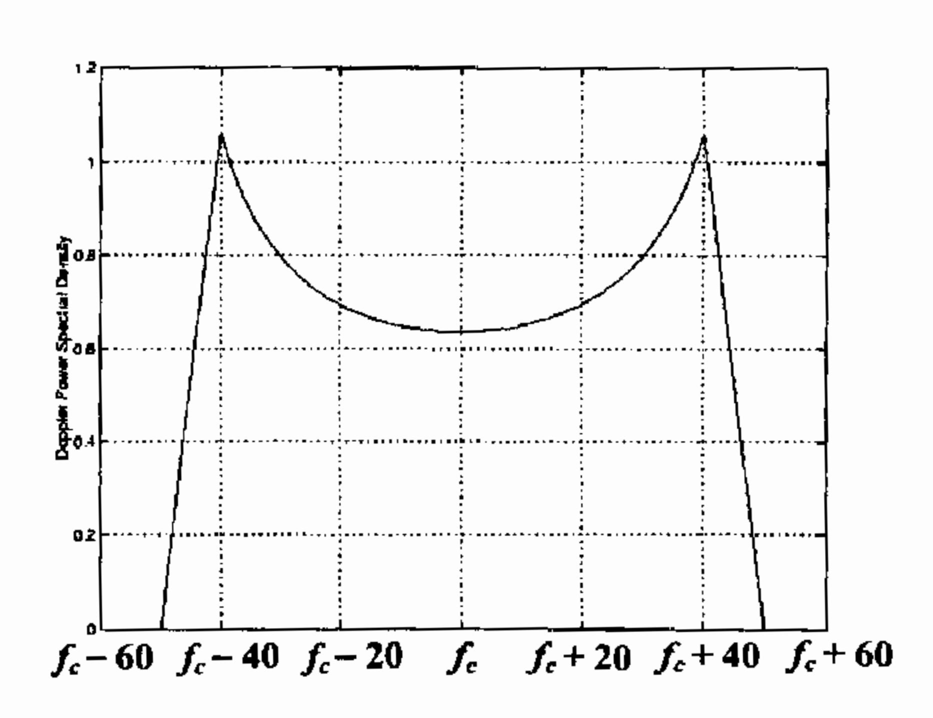
$$f_0 \approx \frac{1}{50\sigma_\tau} = \frac{1}{50\times 0.35\times 10^{-6}} = 57.14 \text{ kHz}$$

(e) The time required to traverse a half wavelength is given as 100 μs, which approximately defines the coherence time. The velocity of the receiver is given as 800 km/hr which corresponds to 222.22 m/s. Therefore, the transmission frequency is obtained from

$$T_0 = \frac{\lambda/2}{V} \qquad \text{or} \qquad \lambda = 2VT_0$$

$$f = \frac{c}{\lambda} = \frac{3\times 10^8 \text{ m/s}}{2\times 222.2 \text{ m/s} \times 10^{-4} \text{ s}} = 6.75 \text{ GHz}$$

15.4 (a) Shown below is a continuous plot of the Doppler power spectral density as plotted in MATLAB ®. The function is even-symmetric about the carrier frequency f_c and is equal to zero outside the range $f_c \pm f_d$.



(b) $S(v)$ has the response that it does at the boundaries due to the sharp upper limit on the Doppler shift produced by a vehicular antenna traveling among the stationary scatterers of the dense scatterer model. The largest magnitude occurs when the scatterer is directly ahead of the moving antenna platform or directly behind it.

(c) For the case where the channel's response to a sinusoid has a correlation greater than 0.5, the relationship between the coherence time and the given Doppler spread of 50 Hz is approximately

$$T_0 \approx \frac{9}{16\pi f_d} = \frac{9}{16\pi \times 50} = 3.6 \text{ ms}$$

15.5

(a) Frequency-selective and fast-fading are characterized by channels having a signal bandwidth that exceeds the channel coherence bandwidth, and a fading rapidity that exceeds the symbol rate. Historically, this was first seen in low-data rate telegraphy channels sent over high-frequency (HF) channels having a narrow coherence bandwidth. Since, we are interested in the fading rapidity as related to the symbol rate, it should be clear that too-slow a signaling rate can be the root cause of fast fading degradation.

(b) Frequency-selective and slow-fading are characterized by channels having a signal bandwidth that exceeds the channel coherence bandwidth, and a symbol rate that exceeds the fading rapidity. An application that generally fits this category is a cellular telephone channel. For example, in the GSM system, signaling is at the rate of 271 ksymbols/s, and a typical value for the channel coherence bandwidth is under 100 kHz. The symbol duration is 3.69 μs, and for a carrier frequency of 900 MHz and a velocity of about 100 km/hr, the coherence time is in the order of about 5-6 ms. Thus there are over a thousand symbols transmitted during the coherence-time interval.

(c) Flat-fading and fast-fading are characterized by channels having a channel coherence bandwidth that exceeds the signal bandwidth, and a fading rapidity that exceeds the symbol rate. An application that can fit this category is a low-data rate system operating in an environment having small multipath delay spread (large channel coherence bandwidth), where the speed of movement results in fast fading. This might be represented by a low-data rate system operating in a fast moving vehicle in a desert environment, or a low-data rate radio on a rapidly moving indoor conveyor belt.

(d) Flat-fading and slow-fading are characterized by channels having a coherence bandwidth that exceeds the signal bandwidth, and a symbol rate that exceeds the fading rapidity. An application that fits this category is an indoor (low-multipath delay spread) high-data rate system. Here, the data rate need not be very large, if we presume that the fastest speed of movement is represented by a person walking.

15.6

(a) The delay spread and the Doppler spread represent functions that are *dual* to each other. Two processes (functions, elements, or systems) are dual to each other if their mathematical relationships are the same even though they are described in terms of different parameters. Here, the Doppler power spectral density, $S(v)$, can be regarded as the dual of the multipath intensity profile, $S(\tau)$, since the former yields knowledge about the frequency spreading of a signal, and the latter yields knowledge about the time spreading of a signal.

(b) Here, we can characterize the duality between the signal time-spreading mechanism as viewed in the frequency domain via the spaced-frequency correlation function, $R(\Delta f)$, and the time-variant mechanism viewed in the time domain via the spaced-time correlation

function, $R(\Delta t)$. $R(\Delta f)$ yields knowledge about the range of frequencies over which two spectral components of a received signal have a strong potential for amplitude and phase correlation. $R(\Delta t)$ yields knowledge about the span of time over which two received signals have a strong potential for amplitude and phase correlation.

However, take note that the dual functions in part (a) are independent of one another, as are those in part (b). See Section 15.4.1.1.

15.7

$$\bar{\tau} = \frac{\sum_k P(\tau_k)\,\tau_k}{\sum_k P(\tau_k)} = \frac{0+1/2\times100+1/2\times200+1/4\times300}{1+1/2+1/2+1/4} = 100\,\text{ns}$$

$$\overline{\tau^2} = \frac{\sum_k P(\tau_k)\,\tau^2{}_k}{\sum_k P(\tau_k)} = \frac{0+1/2\times100^2+1/2\times200^2+1/4\times300^2}{1+1/2+1/2+1/4} = 21{,}111\,\text{ns}^2$$

$$\sigma_\tau = \sqrt{\overline{\tau^2}-\left(\bar{\tau}\right)^2} = \sqrt{21{,}100-100^2} = \sqrt{11{,}111} = 105\,\text{ns}$$

$$f_0 = \frac{1}{5\sigma_\tau} = \frac{1}{5\times105\,\text{ns}} = 1.9\ \text{MHz}$$

Therefore, to avoid using an equalizer, the symbol rate should be less than (considerably less than) 1.9 Msymbols/s.

15.8

Doppler spread $f_d = \dfrac{V}{\lambda}$ $\qquad \lambda = \dfrac{c}{f} = \dfrac{3\times10^8}{1.9\times10^9} = 0.1579\ \text{m}$

$$\Delta\theta/\text{symbol} = \frac{f_d\ \text{Hz}}{R_s\ \text{symbols/s}}\times360^\circ$$

$$5^\circ = \frac{V/0.1579\,\text{m}}{24.3\times10^3\ \text{symbols/s}}\times360^\circ \qquad V = 192\,\text{km/hr}$$

15.9 (a)

Pedestrian: $V = 1$ m/s, $f_0 = 300$ MHz, $\lambda = 1$ m.

Thus, $T_0 \approx \frac{0.5}{f_d} = \frac{0.5}{V/1\,\text{m}} = 0.5\,\text{s}$ $\qquad T_{IL} = 5\,\text{s}$

Pedestrian: $V = 1$ m/s, $f_0 = 3$ GHz, $\lambda = 0.1$ m.

Thus, $T_0 \approx \frac{0.5}{f_d} = \frac{0.5}{V/0.1\,\text{m}} = 0.05\,\text{s}$ $\qquad T_{IL} = 0.5\,\text{s}$

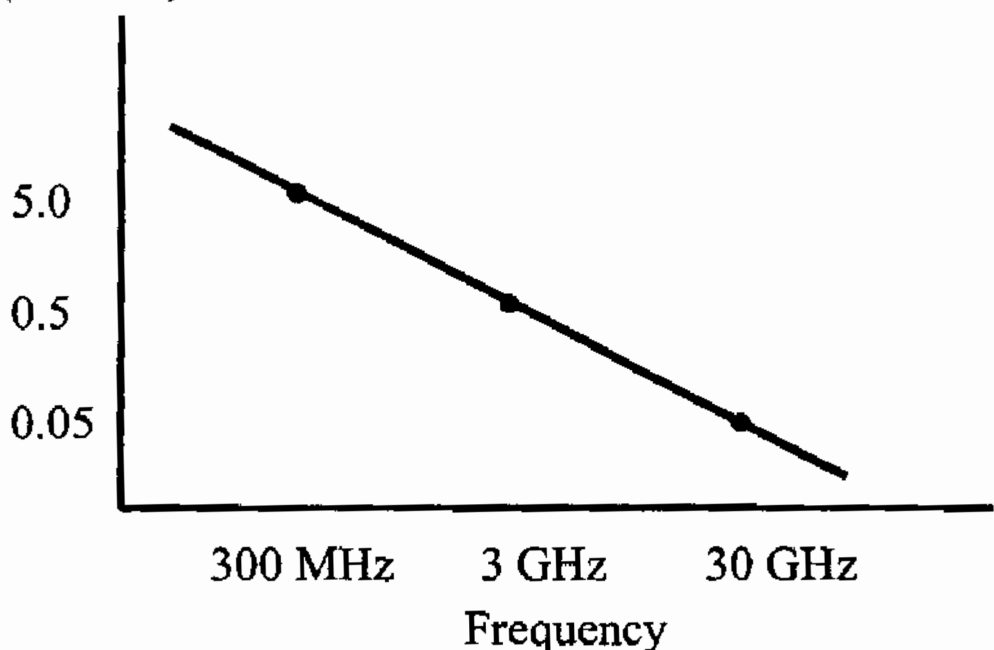
Pedestrian: $V = 1$ m/s, $f_0 = 3$ GHz, $\lambda = 0.1$ m.

Thus, $T_0 \approx \frac{0.5}{f_d} = \frac{0.5}{V/0.1\,\text{m}} = 0.05\,\text{s}$ $\qquad T_{IL} = 0.5\,\text{s}$

Pedestrian: $V = 1$ m/s, $f_0 = 30$ GHz, $\lambda = 0.01$ m.

Thus, $T_0 \approx \frac{0.5}{f_d} = \frac{0.5}{V/0.01\,\text{m}} = 0.005\,\text{s}$ $\qquad T_{IL} = 0.05\,\text{s}$

Meaningful diversity for pedestrian application: T_{IL} versus f



(b) High-speed train: $V = 50$ m/s, $f_0 = 300$ MHz, $\lambda = 1$ m.

$$\text{Thus,}\quad T_0 \approx \frac{0.5}{f_d} = \frac{0.5}{V/1\,\text{m}} = \frac{0.5\,\text{s}}{50/1} = 0.01 \qquad T_{IL} = 0.1\,\text{s}$$

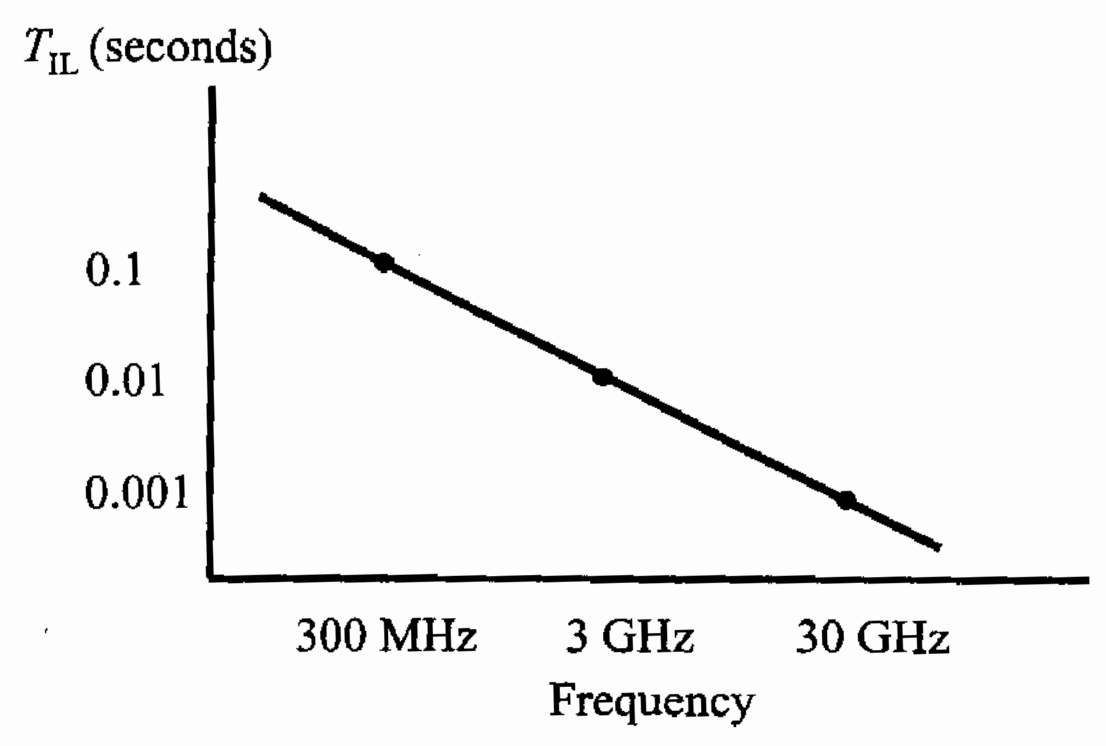
High-speed train: $V = 50$ m/s, $f_0 = 3$ GHz, $\lambda = 0.1$ m.

$$\text{Thus,}\quad T_0 \approx \frac{0.5}{f_d} = \frac{0.5}{V/0.1\,\text{m}} = \frac{0.05\,\text{s}}{50} = 0.001 \qquad T_{IL} = 0.01\,\text{s}$$

High-speed train: $V = 50$ m/s, $f_0 = 30$ GHz, $\lambda = 0.01$ m.

$$\text{Thus,}\quad T_0 \approx \frac{0.5}{f_d} = \frac{0.5}{V/0.01\,\text{m}} = \frac{0.005\,\text{s}}{50} = 0.0001 \qquad T_{IL} = 0.001\,\text{s}$$

Meaningful diversity for high-speed train application: T_{IL} versus f



Conclusions: The faster the speed, and the higher the frequency, the less is the interleaver delay. For a speech application, where we might choose a maximum delay time of 100 ms, the interleaver delay at the transmitter (and the deinterleaver delay at the receiver) may not exceed 50ms. For this example, the pedestrian would only be able to have acceptable diversity for systems whose carrier frequency is 30 GHz or higher.

15.10

For the flat-fading case, where $f_0 > W$ (or $T_m < T_s$), Figure 15.9b shows the usual flat-fading pictorial representation. However, as a mobile radio changes its position, there will be times when the received signal experiences frequency-selective distortion even though $f_0 > W$. This is seen in Figure 15.9c, where the null of the channel's frequency transfer function occurs near the band center of the transmitted signal's spectral density. Thus, even though a channel is categorized as flat fading (based on rms relationships), it can still manifest frequency-selective fading on occasions. It is fair to say that a mobile radio channel, classified as exhibiting flat-fading degradation, cannot exhibit flat fading all of the time. As f_0 becomes much larger than W (or T_m becomes much smaller than T_s), less time will be spent exhibiting the type of condition shown in Figure 15.9c.
By comparison, it should be clear that in Figure 15.9a the fading is independent of the position of the signal band, and frequency-selective fading occurs all the time, not just occasionally.

15.11

$$T_0 \approx \frac{0.5}{f_d} = \frac{0.5}{V/\lambda} \qquad \lambda = \frac{3\times10^8}{1.9\times10^6} = 0.1579\text{ m}$$

$$T_0 \approx \frac{0.5}{(50\text{ m/s})/0.1579\text{ m}} = 1.579\times10^{-3}\text{ s} \qquad \frac{T_0}{4} = 3.9475\times10^{-4}\text{ s}$$

Thus, the training sequence must be received every 3.9475×10^{-4} s. Since the training sequence consists of 20 bits and should not occupy more than 20% of the total bits, then the slowest data rate corresponds to delivering 100 bits in $T_0/4$ s, or $R = \dfrac{100\text{ bits}}{3.9474\times10^{-4}\text{ s}} = 253.3\text{ kbits/s}$.

If the bit rate were any slower, it would require more time than $T_0/4$ s to receive the 20 bit training sequence.

15.12

(a) The condition for frequency selective fading is that $f_0 < W$, i.e., the coherence bandwidth of the channel is less than the signal bandwidth. The channel spacing can be taken to be the maximum signal bandwidth $W = 300$ kHz. The coherence bandwidth is calculated using the following:

$$f_0 \approx \frac{1}{5\sigma_\tau} = \frac{1}{5 \times 300 \times 10^{-9}} = 667 \text{ kHz.}$$

Since $f_0 > W$, the channel is not frequency selective.

(b) We need to check whether the channel coherence bandwidth is less than the signal bandwidth. The signal bandwidth W can be taken to equal the channel spacing which is 1.728 MHz. The coherence bandwidth is equal to:

$$f_0 \approx \frac{1}{5\sigma_\tau} = \frac{1}{5 \times 150 \times 10^{-9}} = 1.33 \text{ MHz}$$

Since $f_0 < W$ we need to include some form of equalization to combat the effects of frequency-selective fading.

15.13

$$T_0 \approx \frac{0.5}{f_d} = \frac{0.5}{V/\lambda} = \frac{\lambda/2}{V} \qquad \lambda = \frac{3 \times 10^8}{10^6} = 0.3\,\text{m}$$

$$= \frac{0.5 \times 0.3}{0.5\,\text{m/s}} = 0.3\,\text{s} \qquad T_{IL} = 3\,\text{s}$$

Such a 3-second interleaver span would not be feasible for speech.

15.14 (a)

Using Equation (15.29), $T_0 \approx \frac{0.5}{f_d} = \frac{0.5}{100\,\text{Hz}} = 0.005\,\text{s}$. Since we desire to keep the interleaver delay down to 100 ms (50 ms at each end), then the desired, then the largest ratio of T_{IL} to T_0 is 0.05 s/0.005 s = 10.

(b) For f_d = 1000 Hz, then T0 = 0.005 s, and the largest ratio of T_{IL} to T_0 is 0.05/0.005 = 100.

15.15 (a)

$$\lambda = \frac{c}{f} = \frac{3\times 10^8}{1.9\times 10^6} = 0.1579\,\text{m} \qquad f_d = \frac{V}{\lambda} = \frac{26.67\,\text{m/s}}{0.1579\,\text{m}} = 168.9\,\text{Hz}$$

The signaling is QPSK. Therefore, the data rate of 200 kbits/s corresponds to a signaling rate of 100 ksymbols/s.

$$\Delta\theta/\text{symbol} = \frac{f_d\ \text{Hz}}{R_s\ \text{symbols/s}} \times 360° = \frac{168.9}{10^5} = 0.61°$$

(b) For a data rate of 100 kbits/s, the QPSK symbol rate is 50 ksymbols/s. Therefore:

$$\Delta\theta/\text{symbol} = \frac{168.9}{5\times 10^4} = 1.22°$$

(c) $f_d = \frac{V}{\lambda} = \frac{13.33\,\text{m/s}}{0.1579\,\text{m}} = 84.4\,\text{Hz} \qquad \Delta\theta/\text{symbol} = \frac{84.4}{5\times 10^4} = 0.61°$

$\Delta\theta/\text{symbol}$ is directly proportional to velocity and inversely proportional to symbol rate.

15.16 (a)

$$f_0 \approx \frac{1}{5\sigma_\tau} = \frac{1}{5\times 10\times 10^{-6}} = 20\,\text{kHz}$$

(b)

$$T_0 \approx \frac{0.5}{f_d} = \frac{0.5}{1\,\text{Hz}} = 0.5\,\text{s}$$

(c) Pulse duration = 1 μs. Thus, $W = R = 10^6$ pulses/s.

$W \gg \frac{1}{T_0}$ Therefore, the channel is slow fading

$f_0 \ll 10^6$ Therefore, the channel is frequency selective.

(d) To mitigate the frequency-selective effects of fading, one could reduce the pulse rate to be less than 20 kpulses/s.

15.17

$$\lambda = \frac{c}{f} = \frac{3\times 10^8}{1.9\times 10^6} = 0.1579\,\text{m} \qquad f_d = \frac{V}{\lambda} = \frac{26.67\,\text{m/s}}{0.1579\,\text{m}} = 168.9\,\text{Hz}$$

$$T_0 \approx \frac{0.5}{f_d} = \frac{0.5}{168.9\,\text{Hz}} = 2.96\,\text{ms} \qquad \frac{1}{T_0} = 337.8$$

Therefore, the slowest signaling rate should be about $100 \times 337.8 \approx 33.8$ ksymbols/s.

15.18

(a) There is a total of 2(4) + 10 + 2(40) = 98 bits per slot. Since we are told that the information is transmitted using QPSK, the number of symbols per slot is equal to 98/2 = 49. The slot duration is equal to:

$$T_{\text{SLOT}} = \frac{49}{33.6\times 10^3} = 1.459\ \text{ms}$$

The receiver speed can be as fast as 100 km/hr, or 27.8 m/s. Given a carrier frequency of 700 MHz, the signal wavelength is:

$$\lambda = \frac{c}{f} = \frac{3\times10^8}{7\times10^8} = 0.428 \text{ m}$$

Next, calculate the time required in order to traverse half a wavelength which is approximately equal to the coherence time:

$$T_0 \approx \frac{\lambda/2}{V} = 7.7 \text{ ms}$$

We have that the coherence time is approximately equal to 5 times the slot duration and so the time required for a substantial change in the fading behavior is relatively long compared to the time duration of a single slot. Thus the midamble should be able to provide accurate information to the receiver regarding the status of the channel, and the system should not suffer the effects of fast fading.

(b) The signal bandwidth is given as 47 kHz, and the rms delay spread is 4 μs. The coherence bandwidth is:

$$f_0 \approx \frac{1}{5\sigma_\tau} = \frac{1}{5\times4.0\times10^{-6}} = 50 \text{ kHz}$$

Since $f_0 > W$, there should be no frequency-selective fading. However, whenever such marginal cases are encountered (f_0 is not much larger than W), an equalizer is always specified.

15.19 We can write the following:

$T_{TOT} = T_{ENC} + T_{MOD} + T_{CHAN} + T_{DEMOD} + 2T_{INT} + T_{DEC}$

where the interleaver delay is included twice, to account for interleaving and de-interleaving. Substituting the values given in Table P15.1, results in the following: $2T_{INT} + T_{DEC} = 340$ ms – 37.3 ms = 302.7 ms.

(a) For an interleaver size of 100 bits, the total interleaving and deinterleaving delay is equal to: $(2\times100)/(19.2\times10^3) = 10.4$ ms. Therefore the allowable time to perform decoding is equal to $T_{DEC} = 302.7 - 10.4$ ms = 292.3 ms. We are given that $T_{DEC} = (2\times10^8)/f_{clk}$ ms. Thus, the minimum decoder clock speed is approximately equal to 684 kHz.

(b) For an interleaver size of 1000, the interleaving and deinterleaving delay is equal to $(2\times1000)/(19.2\times10^3) = 104$ msecs. Therefore the allowable time to perform decoding is equal to $T_{DEC} = 302.7 - 104$ ms = 198.7 ms. The mimimum decoder clock speed is approximately equal to 1 MHz.

(c) For an interleaver size of 2850, the interleaving and deinterleaving delay is equal to $(2\times2850)/(19.2\times10^3) = 297$ ms. Therefore the allowable time to perform decoding is equal to $T_{DEC} = 302.7 - 297$ ms = 5.7 ms, and the mimimum decoder clock speed is approximately equal to 35 MHz.

(d) As the interleaver size increases, the decoder clock speed increases. However we note that increasing the interleaver size by a factor of 10 from 100 to 1000 causes the decoder clock speed to increase by a factor of $1/0.684 \approx 1.5$. However increasing the interleaver size by a further factor of 2.85 causes the decoder clock speed to jump up by a factor of ≈ 35. Therefore there is a trade-off [illegible] accounted for between a larger interleaver size [illegible]sults in better BER performance and increasing [illegible]ich results in the use of more advanced and [illegible]re technologies.

15.20

We start with our desire that $f_0 > W > f_d$, and we assume that the signal bandwidth, W, is approximately equal to the signaling rate, $1/T_s$. As a first estimate of this signaling rate, we use the geometric mean between f_0 and f_d.

$$T_0 \approx \frac{0.5}{f_d} = \frac{0.5}{V/\lambda} \qquad \lambda = \frac{3\times10^8}{3\times10^9} = 0.1\,\text{m}$$

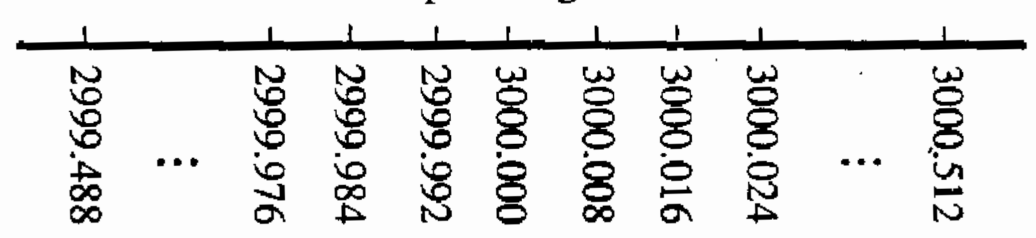
$$= \frac{0.5}{(22.22\,\text{m/s})/0.1\,\text{m}} = 2.25\times10^{-3}\,\text{s}$$

$$f_d \approx \frac{1}{T_0} = \frac{1}{2.25\times10^{-3}} = 444\,\text{Hz}$$

The geometric mean between the given f_0 of 100 kHz and f_d of 444 Hz is 6663 symbols/s. For each subcarrier, let us choose a symbol rate of 8000 symbols/s. Then an OFDM plan would contain

$$\text{Number of subcarriers} = \frac{1024\times10^3\ \text{symbols/s}}{8000\ \text{symbols/s/subcarrier}} = 128\ \text{subcarriers}$$

A subcarrier plan might look like this:



Frequency (MHz)

15.21

The multipath delay is: $\tau = \frac{d}{c} = \frac{120\,\text{m}}{3\times10^8\,\text{m/s}} = 0.4\times10^{-6}\,\text{s}$

Hence, the required chip rate is: $R_{ch} \geq \frac{1}{\tau} = 2.5\,\text{Mchips/s}$

15.22

DS/SS systems can typically mitigate channel-induced ISI at the symbol level – not at the chip level. However, with a reasonable amount of processing gain, the DS/SS system is robust enough to withstand the interchip interference.

A DDS/SS system with a processing gain of 1000 transmits 1000 chips per bit. To get an intuitive feeling about the robustness of such a system, make believe that detection at the receiver is performed on each chip (it doesn't actually work that way – the correlator with its PN reference performs product-integration of the received sequence of chips with the sequence of PN code reference chips, and during a symbol interval accumulates a signal which is then compared to a threshold). But for a moment, imagine a binary scheme, where an individual decision is made on each chip, and then some voting logic was used. In that case, do you see, that 499 of those decisions can be wrong and the detector's response for that bit would still be correct? The spread-spectrum processing gain allows the system to endure such interference at the chip level. No other equalization is deemed necessary.

15.23

CDMA, either direct sequence (DS) or frequency hopping (FH) can provide mitigation against the effects of frequency-selective fading. In the case of DS, the spread spectrum system, effectively eliminates the multipath interference by virtue of its code-correlation receiver. (See Section 15.5.1.). FH can also be used as a technique to mitigate the distortion caused by frequency-selective fading, provided the hopping rate is at least equal to the symbol rate. Compared to DS/SS, mitigation takes place through a different mechanism. FH receivers avoid the degradation effects due to multipath by rapidly changing in the transmitter carrier-frequency band, thus avoiding the interference by changing the receiver band position before the arrival of the multipath signal.

15.23 (cont'd.)

TDMA can most naturally provide mitigation against fast fading. This comes about because TDMA systems are burst systems. Users transmit in bursts when their assigned slot appears. Thus, the transmission rate in a TDMA system is much higher than would ordinarily be needed for sending the same information on a dedicated channel. For example, in GSM the signaling rate for voice signals is 271 ksymbols/s. Since fast-fading occurs whenever the symbol rate is less than the fading rate, there is a natural protection agains this type of degradation when using faster signaling.

f

15.24

We first calculate the total signal envelope in terms of the voltage gains, G_i, which is given by:

$$r_M = \sum_{i=1}^{M} G_i r_i \quad \text{where } M = 4$$

$$= (0.5)\,(0.87) + (0.8)\,(1.21) + (1.0)\,(0.66) + (0.8)\,(1.90) = 3.583 \text{ volts}$$

Since each signal is received with its own demodulator, then we next can calculate the total noise power given by:

$$N_T = N \sum_{i=1}^{M} G_i^2$$

$$= 0.25\,(0.5^2 + 0.8^2 + 1.0^2 + 0.8^2) = 2.53$$

For the signal-to-noise ration, γ, we can now write,

$$\gamma_M = \frac{1}{2}\frac{r_M^2}{N_T} = \frac{3.583^2}{2(2.53)} = 2.537$$

where the factor of ½ stems from the fact that the total average normalized power of a bandpass waveform can be shown to equal ½ of the average of the envelope magnitude-square [1, 10].

(b) For the case where $G_i = r_i^2 / N$ and the SNR out of the diversity combiner is the sum of the SNRs in each branch, the sum of the individual SNRs is

$$\sum_{i=1}^{M} \frac{r_i^2}{2N} = \left(\frac{0.87^2 + 1.21^2 + 0.66^2 + 1.90^2}{2\,(0.25)} \right) = 12.53$$

15.25

(a) To solve for *M*, we need to rearrange the following expression:

$$P(\gamma_1, \gamma_2, \ldots, \gamma_M \leq \gamma) = \left[1 - \exp\left(-\frac{\gamma}{\Gamma} \right) \right]^M$$

We have that Γ = 15 dB and the threshold γ = 5 dB. Thus, $\gamma/\Gamma = 0.1$, and we have the following:

$$10^{-4} = \left[1 - \exp(-0.1) \right]^M$$

On re-arranging

$$M = \frac{\ln 10^{-4}}{\ln\left[1 - \exp(-0.1) \right]} = 3.91$$

Thus we require at least 4 branches in order to meet the above specification.

(b) We use the following:

$$P(\gamma_i > \gamma) = 1 - \left[1 - \exp\left(-\frac{\gamma}{\Gamma} \right) \right]^M$$ which we calculate, using $M = 4$

$$P(\gamma_i > \gamma) = 1 - \left[1 - \exp(-0.1) \right]^4 = 0.9999179$$

15.26

(a) With selection diversity we determine the instantaneous value of SNR for each branch which we assume is used in order to make the selection process. The SNR for branch i is given by:

$$\mathrm{SNR}_i = \frac{r_i^2}{2N}$$

and since the average noise power is given to be the same for both branches, we can make our selection based on the branch whose signal has the maximum amplitude value squared. In this case the following would be selected:

[B1, B1, B2, B1, B2, B1, B2, B2, B2, B1]

We assume that the selection operates on an instantaneous basis from one time interval to the next.

(b) With feedback diversity the signals are scanned in a fixed sequence until one is found which is above a predetermined threshold. This signal is received until it falls below the threshold value upon which the scanning process is re-initiated. We can firstly calculate the voltage level for which the threshold corresponds to. We know that:

$$5 = 10\log\left(\frac{r_i^2}{2N}\right)$$

which upon rearranging and introducing the value $N = 0.25$ gives $r_i = \pm 1.257$. Since both branches have the same average noise power, we just need to ensure that we scan until a branch which has $|r_i| = 1.257$ is found. Assume that we begin scanning from Branch 1. In this case, the following would be selected:

[B1, B1, B1, B1, B2, B2, B2, B2, B2, B1]

It is worth noting that with selection diversity, an improvement in the SNR can be achieved without great complexity in the receiver. To achieve this kind of diversity, it is only necessary to implement an antenna switch and a monitoring algorithm (to determine if and when switching is required). Feedback diversity has the advantage that it is also very easy to implement, although it does not provide the diversity advantages achievable through the use of more complex techniques, which follow.

(c) With maximum ratio combining, we need to take into account the respective gains of each branch that has been supplied. The following needs to be calculated for each time interval k:

The signal envelope given by: $r_M = \sum_{i=1}^{M} G_i r_i$

The total noise power given by: $N_T = N \sum_{i=1}^{M} G_i^2$

And finally the SNR is: $\gamma_M = \dfrac{r_M^2}{2N_T}$

where the summations given above are over two elements. If we assume we begin at time $k = 1$ and finish at time $k = 10$, the following is obtained:

$$\underline{k=1}: \; \gamma_M = \frac{(1.2)^2(1.85)^2 + (1.4)^2(1.67)^2}{2(0.25)\left[(1.2)^2 + (1.4)^2\right]} = \frac{10.39}{1.7} = 6.11$$

$$\underline{k=2}: \; \gamma_M = \frac{(1.2)^2(1.91)^2 + (1.4)^2(1.69)^2}{2(0.25)\left[(1.2)^2 + (1.4)^2\right]} = \frac{10.85}{1.7} = 6.38$$

$$\underline{k=3}: \; \gamma_M = \frac{(1.2)^2(-1.31)^2 + (1.4)^2(-2.13)^2}{2(0.25)\left[(1.2)^2 + (1.4)^2\right]} = \frac{11.36}{1.7} = 6.68$$

$\underline{k=4}$: $\gamma_M = \dfrac{(1.2)^2(-1.58)^2+(1.4)^2(-1.26)^2}{2(0.25)\left[(1.2)^2+(1.4)^2\right]} = \dfrac{6.71}{1.7} = 3.95$

$\underline{k=5}$: $\gamma_M = \dfrac{(1.2)^2(1.21)^2+(1.4)^2(1.74)^2}{2(0.25)\left[(1.2)^2+(1.4)^2\right]} = \dfrac{8.04}{1.7} = 4.73$

$\underline{k=6}$: $\gamma_M = \dfrac{(1.2)^2(1.93)^2+(1.4)^2(1.76)^2}{2(0.25)\left[(1.2)^2+(1.4)^2\right]} = \dfrac{11.44}{1.7} = 6.73$

$\underline{k=7}$: $\gamma_M = \dfrac{(1.2)^2(1.11)^2+(1.4)^2(1.29)^2}{2(0.25)\left[(1.2)^2+(1.4)^2\right]} = \dfrac{5.04}{1.7} = 2.96$

$\underline{k=8}$: $\gamma_M = \dfrac{(1.2)^2(-1.67)^2+(1.4)^2(-1.93)^2}{2(0.25)\left[(1.2)^2+(1.4)^2\right]} = \dfrac{11.32}{1.7} = 6.66$

$\underline{k=9}$: $\gamma_M = \dfrac{(1.2)^2(2.13)^2+(1.4)^2(2.31)^2}{2(0.25)\left[(1.2)^2+(1.4)^2\right]} = \dfrac{16.99}{1.7} = 9.99$

$\underline{k=10}$: $\gamma_M = \dfrac{(1.2)^2(-2.25)^2+(1.4)^2(-1.08)^2}{2(0.25)\left[(1.2)^2+(1.4)^2\right]} = \dfrac{9.58}{1.7} = 5.63$

(d) For equal gain combining we have the same as above, except that now the gain in all branches is set to unity. We obtain the following:

$\underline{k=1:}$ $\gamma_M = \dfrac{(1.85)^2+(1.67)^2}{2(0.25)\left[(1.2)^2+(1.4)^2\right]} = \dfrac{6.21}{1.7} = 3.65$

$\underline{k=2}$: $\gamma_M = \dfrac{(1.91)^2+(1.69)^2}{2(0.25)\left[(1.2)^2+(1.4)^2\right]} = \dfrac{6.50}{1.7} = 3.83$

$\underline{k=3}$: $\gamma_M = \dfrac{(-1.31)^2+(-2.13)^2}{2(0.25)\left[(1.2)^2+(1.4)^2\right]} = \dfrac{6.25}{1.7} = 3.68$

$$\underline{k=4}: \quad \gamma_M = \frac{(-1.58)^2 + (-1.26)^2}{2(0.25)\left[(1.2)^2 + (1.4)^2\right]} = \frac{4.08}{1.7} = 2.40$$

$$\underline{k=5}: \quad \gamma_M = \frac{(1.21)^2 + (1.74)^2}{2(0.25)\left[(1.2)^2 + (1.4)^2\right]} = \frac{4.49}{1.7} = 2.64$$

$$\underline{k=6}: \quad \gamma_M = \frac{(1.93)^2 + (1.76)^2}{2(0.25)\left[(1.2)^2 + (1.4)^2\right]} = \frac{6.82}{1.7} = 4.01$$

$$\underline{k=7}: \quad \gamma_M = \frac{(1.11)^2 + (1.29)^2}{2(0.25)\left[(1.2)^2 + (1.4)^2\right]} = \frac{2.90}{1.7} = 1.70$$

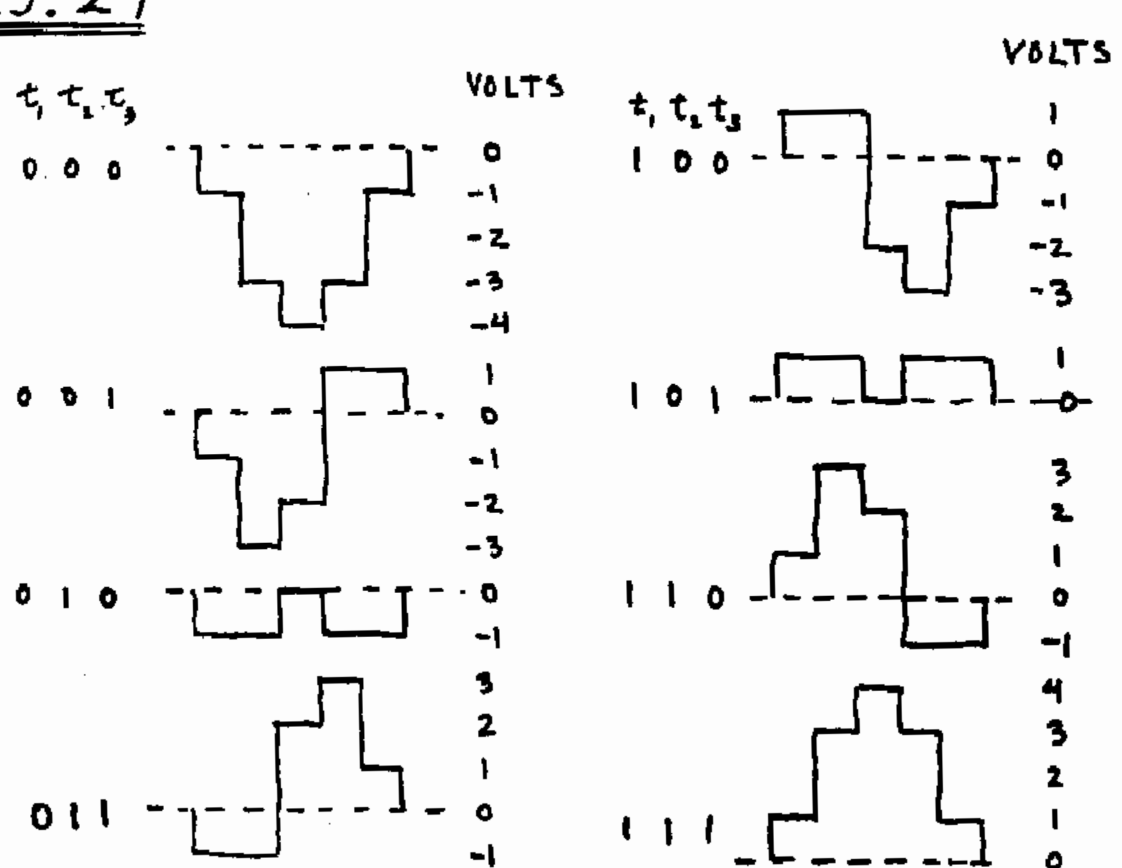
$$\underline{k=8}: \quad \gamma_M = \frac{(-1.67)^2 + (-1.93)^2}{2(0.25)\left[(1.2)^2 + (1.4)^2\right]} = \frac{3.83}{1.7} = 2.25$$

$$\underline{k=9}: \quad \gamma_M = \frac{(2.13)^2 + (2.31)^2}{2(0.25)\left[(1.2)^2 + (1.4)^2\right]} = \frac{9.87}{1.7} = 5.81$$

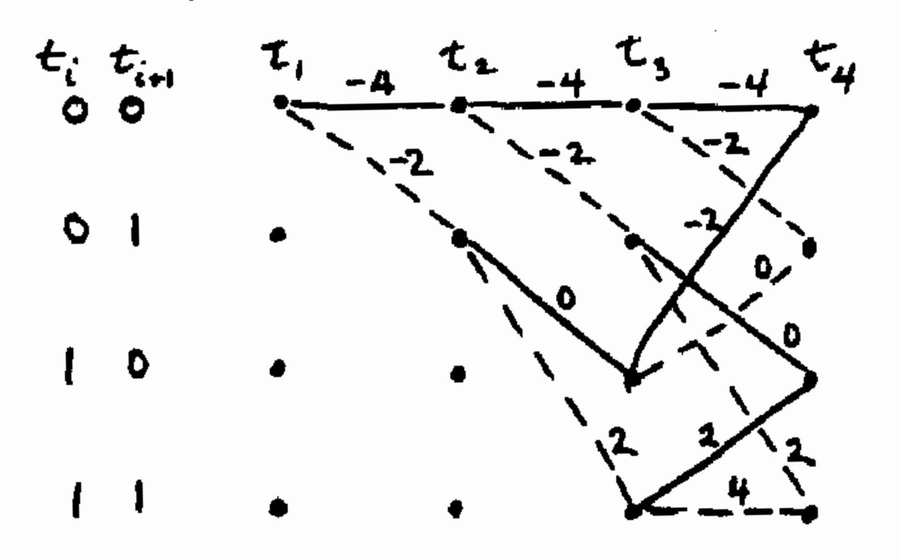
$$\underline{k=10}: \quad \gamma_M = \frac{(-2.25)^2 + (-1.08)^2}{2(0.25)\left[(1.2)^2 + (1.4)^2\right]} = \frac{6.23}{1.7} = 3.66$$

Both maximal-ratio and equal-gain combining typically provide better performance than selection and feedback diversity since at any one time, when a diversity calculation is made, all the information available in ALL the branches is utilized. Equal-gain combining has slightly worse performance than that which is available with maximal-ratio combining, since maximal-ratio combining utilizes variable weights which optimize the maximum available SNR. This can be seen from the solutions to parts (c) and (d), since the SNR at each time instant obtained with maximal-ratio combining is superior to that obtained with equal-gain combining. It is worth noting however that equal gain combining still manifests better SNR performance than selection and feedback diversity since, it still makes use of information available in all branches.

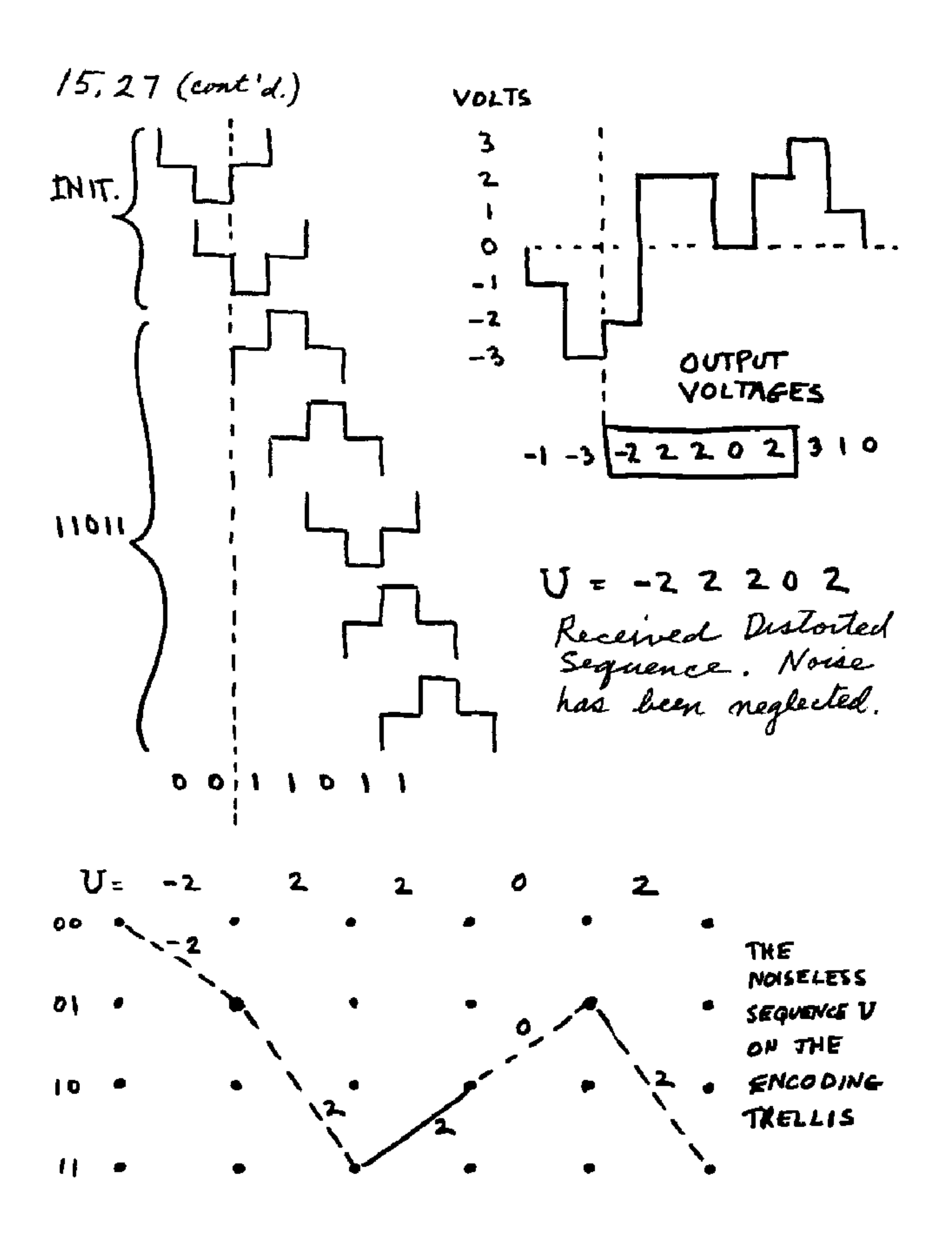
15.27

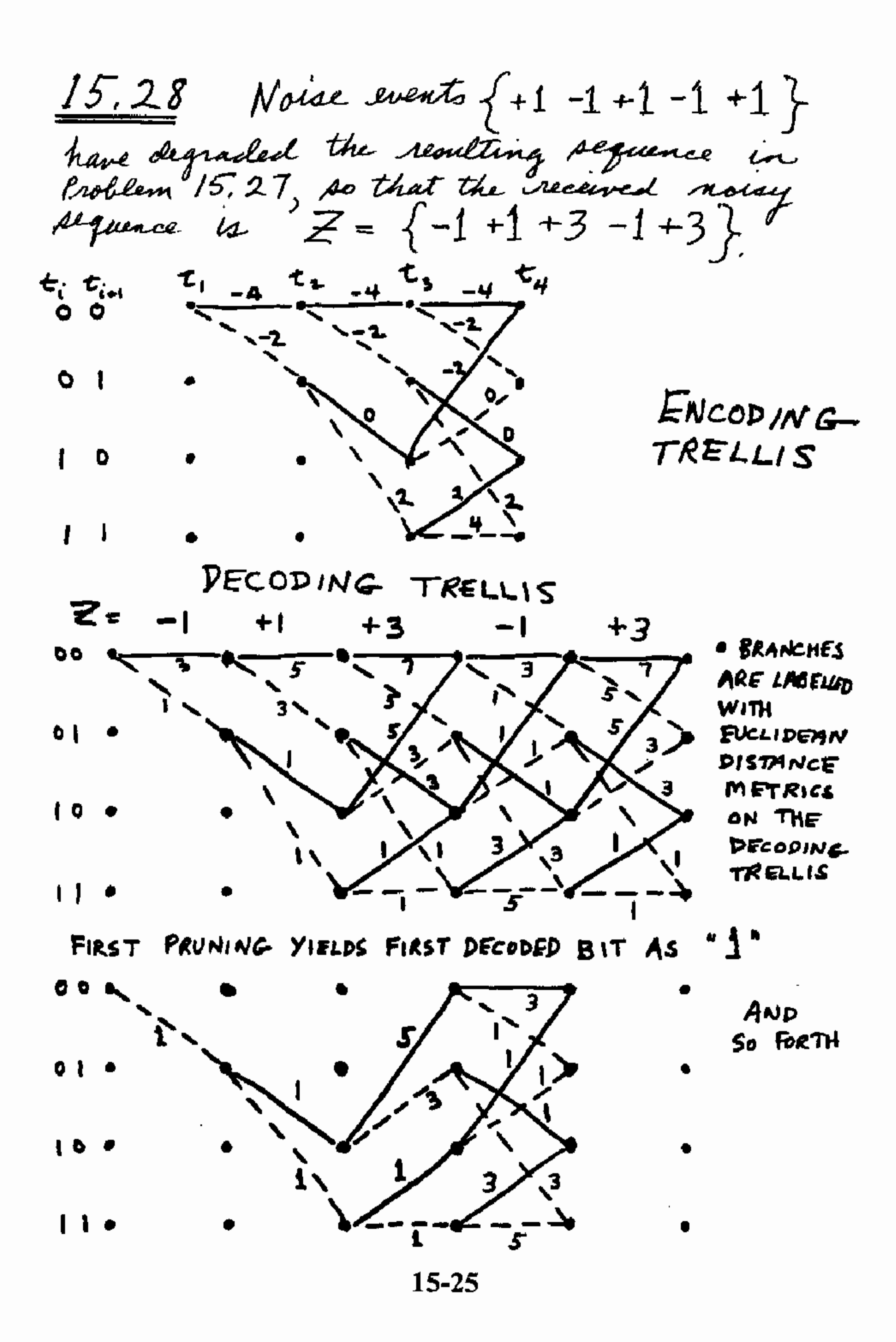


The state is represented by $L-1 = 2$ prior bits where the leftmost is the earliest bit, and the trellis diagram below represents an encoding trellis. The trellis starts at time t_1 (after initialization into the 00 state). On each trellis transition is written the voltage value that results from that transition.



15.27 (cont'd.)


VOLTS
3
2
1
0
-1
-2
-3
INIT.
11011
0 0 1 1 0 1 1
OUTPUT VOLTAGES
-1 -3 -2 2 2 0 2 3 1 0
U = -2 2 2 0 2
Received Distorted Sequence. Noise has been neglected.
U = -2 2 2 0 2
00
01
10
11
-2
2
2
0
2
THE NOISELESS SEQUENCE U ON THE ENCODING TRELLIS



15.28 Noise events $\{+1\ -1\ +1\ -1\ +1\}$ have degraded the resulting sequence in Problem 15.27, so that the received noisy sequence is $Z = \{-1\ +1\ +3\ -1\ +3\}$.

15.28 (cont'd.)

Problems 15.27 and 15.28 illustrate a technique for mitigating the "smearing" effects of ISI. Because the Viterbi decoding algorithm is used, the technique is often referred to as Viterbi equalization. In Problem 15.27, the ISI corresponds to a channel having memory, in that each signaling interval represents the superposition of several symbol components. Whenever a signal is constrained by a memory of the past, the signaling scheme can be referred to as a "finite-state machine," and a trellis diagram is a simple way to describe it. Problem 15.27 lays out the nature of the ISI caused either by circuitry or by a multipath channel (or both), and asks for a description of the resulting smeared waveform and its trellis-diagram characterization (encoding trellis). The only difference between this trellis and the ones described in Chapter 7, is that here we show channel-waveform voltage values rather than channel-bit values on the trellis transitions. In Problem 15.28, after noise has been added to the distorted sequence, we can estimate the original message sequence with the use of a decoding trellis. This follows the same Viterbi algorithm described in Chapter 7. The only difference is that here the metric placed on each trellis transition is the voltage difference between the signal that was received and the *noiseless* signal that would have been received had the encoder made the transition in question. The rest of the signal processing is exactly the same as in the decoding of convolutionally encoded bits. Once the trellis is pruned, so that a "common stem" appears, a bit-decoding can take place, where dashed lines and solid lines represent 1 and 0 respectively.

15.29

(a) The bit rate is equal to the symbol rate because the modulation is binary (BPSK). The bit period is equal to $1/(160\times10^3) = 6.25$ μs. The amount of dispersion in the signal is equal to 25 μs, and therefore the Viterbi equalizer requires a memory corresponding to approximately (25/6.25) 4 bit intervals.

(b) For a doubling of the bit rate and assuming BPSK modulation is still used, the bit period is now equal to $1/(2\times160\times10^3) = 3.125$ μs. The Viterbi equalizer now requires a memory span of $(25/3.125) = 8$ bit intervals.